PROPERTIES OF THE SAMPLE AUTOCORRELATIONS IN AUTOREGRESSIVE STOCHASTIC VOLATILITY MODELS
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Abstract
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Keywords: Heteroscedastic time series; correlogram; non-linear transformations; asymptotic distribution.

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Properties of the Sample Autocorrelations in AutoRegressive
Stochastic Volatility Models

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Abstract: Time series generated by Stochastic Volatility (SV) processes are uncorrelated although not independent. This has consequences on the properties of the sample autocorrelations. In this paper, we analyse the asymptotic and finite sample properties of the correlogram of series generated by SV processes. It is shown that the usual uncorrelatedness tests could be misleading. The properties of the correlogram of the log-squared series, often used as a diagnostic of conditional heteroscedasticity, are also analysed. It is proven that the more persistent and the larger the variance of volatility, the larger the negative bias of the sample autocorrelations of that series.

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1. INTRODUCTION

High frequency financial time series are often characterised by being serially uncorrelated although not independent. They exhibit time varying volatilities that imply autcorrelated squared observations. The AutoRegressive Stochastic Volatility (ARSV) model was proposed by Taylor (1986) to represent the dynamic evolution of volatility which is specified as an unobservable process, $\sigma$, the logarithm of which is modelled as a linear autoregressive process. Stochastic Volatility (SV) processes have the advantage of being very flexible to represent the stochastic properties of real time series with conditional heteroscedasticity; see Ghysels, Harvey and Renault (1996) for a detailed review. Moreover, their dynamic properties are easily obtained from the properties of the process generating the variance component. ARSV processes generate non-Gaussian and uncorrelated series. However, they imply correlations in the squared observations with a structure similar to an ARMA(1,1) process; Taylor (1986, p.75). Therefore, knowledge of the properties of the correlogram is important for modelling temporal dependence in volatility. In this paper, we investigate the properties of the sample autocorrelations of time series generated by ARSV processes. We also consider the properties of the correlogram of the log-squared observations, that is often used to test for the presence of time varying volatility.

The sample autocorrelations of linear time series are known to be downward biased estimators of their population counterparts. Fuller (1996, chap.6) proposes approximated formulas to compute this bias in finite samples. Regarding the asymptotic distribution of the sample autocorrelations, Anderson and Walker (1964) derive the main result for linear models with independent and identically distributed (I.I.D.) disturbances. Hannan
and Heyde (1972) and Anderson (1992) generalise this result to the case where the disturbances are a martingale difference. Further results can be found in Anderson (1971, chap.8) and Brockwell and Davies (1991, chap.7).

With respect to heteroscedastic time series, Milhoj (1985) studies the effect of conditional heteroscedasticity on the asymptotic properties of the correlogram of series generated by AutoRegressive Conditionally Heteroscedasticity (ARCH) processes. Based upon these results, Diebold (1988) shows that the usual Barlett confidence bands and the Box-Pierce test can be misleading for such processes. Bollerslev (1988) and He and Terasvirta (1999) also show, by Monte Carlo experiments, that in GARCH (Generalized ARCH) processes, the correlogram of the squared series exhibits severe downwards bias.

The behaviour of the correlogram of series generated by SV processes has not received much attention in the literature. Taylor (1986, chap. 5) proposes an estimate of the variance of the sample autocorrelations for ARSV processes and suggests that using the standard methodology for those processes will make the usual uncorrelatedness tests unreliable. The objective of this paper is to analyse the asymptotic and finite sample properties of the sample autocorrelations of series generated by the ARSV process and evaluate their consequences on the tests involving these autocorrelations. We will also derive the properties of the correlogram of the log-squared observations.

The outline of the paper is as follows. In section 2, the population moments and correlation structure of ARSV processes are presented. In section 3, we analyse the asymptotic and finite sample properties of the correlogram of series generated by ARSV processes and the properties of the usual tests for uncorrelatedness. We prove that the asymptotic distribution is not the usual one and we propose a correction for volatility. We also show, by means of Monte Carlo experiments, that this correction works
appropriately in finite samples. In section 4, the asymptotic distributions of the sample autocovariances and the sample autocorrelations of log-squared observations are established. The asymptotic results are accompanied with the results of a simulation study to assess their small sample performance. Section 5 includes the empirical analysis of a series of daily returns of the Spanish Stock Market index IBEX-35. Finally, section 6 contains a few concluding remarks.

2. STATISTICAL PROPERTIES OF SV PROCESSES

SV processes represent the series of interest, $y_t$, as the product of a sequence of I.I.D. random variables, $\varepsilon_t$, and a non-observable stochastic process, $\sigma_t$, called volatility. In the simplest set up, the logarithm of the volatility, $h_t = \log(\sigma_t^2)$, is generated by an autoregressive process of order one. The resultant SV process, called ARSV, is given by:

$$y_t = \sigma_0 \sigma_t \varepsilon_t, \quad \sigma_t = \exp(h_t/2), \quad (1a)$$

$$h_t = \phi h_{t-1} + \eta_t \quad (1b)$$

where $\sigma_0$ is a scale factor that removes the need of including a constant term in equation (1b), $\phi$ is an unknown parameter that satisfies the condition $|\phi|<1$, $\varepsilon_t$ is an independent Gaussian process with zero mean and unit variance, i.e., $\varepsilon_t$ is NID(0,1), and $\eta_t$ is NID(0,$\sigma_\eta^2$), generated independently of $\varepsilon_s$ for all $t,s$. Although the assumption of Gaussianity for $\eta_t$ can seem *ad hoc* at first sight, Andersen, Bollerslev, Diebold and Labys (1999) show that the log-volatility distribution can be well approximated by a Normal distribution. On the other hand, the assumption of Gaussianity for $\varepsilon_t$ is not as restrictive as it is in GARCH processes. In fact, the ARSV process with $\varepsilon_t$ being Normal
still allows simultaneously for excess kurtosis in the marginal distribution of \( y_t \) and low but persistent autocorrelations in the squares; see Carnero, Peña and Ruiz (2001).

The statistical properties of the basic ARSV process in (1) are given, for example, by Harvey (1993, chap.8) who shows that \( y_t \) is a martingale difference. Therefore, \( E(y_t)=0 \) and \( \text{Cov}(y_t, y_s)=0 \) for all \( t \neq s \). As \( \epsilon_t \) is always stationary, the condition \(|\phi|<1\) is enough for \( y_t \) to be stationary with finite variance. Moreover, using the properties of the logNormal distribution, the second and fourth order moments of \( y_t \) are obtained as follows:

\[
\text{Var}(y_t)=\sigma^2 \, E(\epsilon_t^2) \text{E}[\exp(h_t)]=\sigma^2 \exp(\sigma^2_\eta/2) \quad (2)
\]

\[
E(y_t^4)=\sigma^4 \, E(\epsilon_t^4) \text{E}[\exp(2h_t)]=3 \, \sigma^4 \exp(2 \, \sigma^2_\eta) \quad (3)
\]

where \( \sigma^2_\eta \) is the variance of \( h_t \), given by \( \sigma^2_\eta = \sigma^2/(1-\phi^2) \). Therefore, \(|\phi|<1\) is also the condition for the fourth order moment of \( y_t \) to exist and to be finite.

Although the series \( y_t \) is uncorrelated, it is not an independent sequence. The dynamics of the series appear in the squared observations. The autocovariance function of \( y_t^2 \) is given by:

\[
\gamma_{y^2}(k)=\text{Cov}(y_t^2, y_{t+k}^2)=\sigma^4 \, \exp(\sigma^2_\eta) \{ \exp[\gamma_h(k)]-1 \}, \text{ for } k \geq 1,
\]

(4)

where \( \gamma_h(k) \) is the autocovariance function of the process \( h_t \) given by:

\[
\gamma_h(k)=\sigma^2_\eta \, \phi^k, \quad \text{for } k \geq 1.
\]

(5)

Taylor (1986, pp. 74-75) notes that if \( \sigma^2_\eta \) is small and/or \( \phi \) is close to one, the autocorrelation function of \( y_t^2 \) is very close to that of an ARMA(1,1) process.

As Harvey, Ruiz and Shephard (1994) point out, the dynamic properties of the ARSV process appear more clearly in the series \( \log(y_t^2) \). Taking logarithms of squares in (1a)
gives the following linear representation of the ARSV process:

\[ x_t = \log(y_t^\gamma) = \mu + h_t + \xi_t \quad (6a) \]

\[ h_t = \phi h_{t-1} + \eta_t \quad (6b) \]

where \( \mu = \log(\sigma^2 + E[\log(\epsilon_t^2)]) \) and \( \xi_t = \log(\epsilon_t^2) - E[\log(\epsilon_t^2)] \) is a non-Gaussian, zero mean, white noise process whose properties depend on the distribution of \( \epsilon_t \). Given that \( \epsilon_t \) is \( N(0,1) \), we have \( E[\log(\epsilon_t^2)] = -1.27, \sigma^2 = \text{Var}(\xi_t) = \pi^2/2 \approx 4.93 \) and \( E(\xi_t^4) = 3\sigma^4 + \pi^4 \); see Abramowitz and Stegun (1970, p. 943).

The series \( x_t \) in equation (6a) is the sum of two independent processes, a stationary AR(1) process, \( h_t \), plus a non-Gaussian white noise, \( \xi_t \). This decomposition makes it easier to obtain the variance, the autocovariance and the autocorrelation function (ACF) of \( x_t \), namely:

\[ \gamma(0) = \text{Var}(x_t) = \sigma_h^2 + \sigma_\xi^2 \quad (7) \]

\[ \gamma(k) = \text{Cov}(x_t, x_{t+k}) = \text{Cov}(h_t + \xi_t, h_{t+k} + \xi_{t+k}) = \gamma_h(k) = \sigma_h^2 \phi^k, \text{ for } k \geq 1 \quad (8) \]

\[ \rho(k) = \frac{\gamma(k)}{\gamma(0)} = \frac{\gamma_h(k)}{\sigma_h^2 + \sigma_\xi^2} = \rho_h(k) \frac{\sigma_h^2}{\sigma_h^2 + \sigma_\xi^2} = \phi^k \frac{\sigma_h^2}{\sigma_h^2 + \sigma_\xi^2}, \text{ for } k \geq 1 \quad (9) \]

Expression (9) states that the ACF of \( x_t \) is proportional to the ACF of \( h_t \). Therefore, the shape of the latter is carried over to the former, except by the factor of proportionality that is less than one.

From (6), it is possible to obtain the following alternative expression of \( x_t \):

\[ (1-\phi L)(x_t-\mu) = \eta_t + \xi_t - \phi \xi_{t-1} \]

From this representation, it is immediate to show that the ACF of \( (1-\phi L)(x_t-\mu) \) exhibits the cut-off at lag one characteristic of the MA(1) process. Therefore, the reduced form of
where $\phi = \frac{- (q + 1 + \phi^2) + \sqrt{(q + 1 + \phi^2)^2 - 4\phi^2}}{2\phi}$ with $q = \sigma^2_h / \sigma^2_{\xi}$, $(\phi, \theta)$ are constrained to be in $\{0 < \phi < 1, -1 < \theta < 0\}$ or $\{-1 < \phi < 0, 0 < \theta < 1\}$ and $E(z_t^2) = -(\phi/\theta) \sigma^2_{\xi}$. Nerlove, Grether and Carvalho (1979, p.73) point out that as $\xi_t$ is not Gaussian, the disturbance $z_t$ of model (10) is uncorrelated but not independent. As we will see later, this will have important implications for the asymptotic properties of the sample autocorrelations of the series $x_t$.

The fourth-order moment of $x_t$ can also be obtained from equation (6a) as follows:

$$E(x_t - \mu)^4 = E(h_t + \xi_t)^4 = E(h_t^4) + 6\sigma^2_h \sigma^2_{\xi} + E(\xi_t^4) = 3[\gamma(0)]^2 + \pi^4$$

where $\gamma(0)$ is the variance of $x_t$ in (7). Therefore, the fourth-order moment of $x_t$ exists and is finite if $|\phi|<1$.

We now turn to the evaluation of the fourth-order cumulant of $x_t$, denoted by $\kappa_x(s,r,q)$. Taking into account that $x_t$ is the sum of two mutually independent processes, $h_t$ and $\xi_t$, it turns out that:

$$\kappa_x(s,r,q) = \kappa_h(s,r,q) + \kappa_\xi(s,r,q)$$

where $\kappa_h(s,r,q)$ and $\kappa_\xi(s,r,q)$ are the fourth-order cumulants of $h_t$ and $\xi_t$, respectively; see Brillinger (1981, p.19). Since $h_t$ is a stationary zero-mean Gaussian process, all their fourth-order cumulants are zero (Anderson 1971, p. 444). Moreover, since $\xi_t$ is an independent sequence, all their fourth-order cumulants are also zero, except $\kappa_\xi(0,0,0) = E(\xi_t^4) - 3\sigma^2_{\xi} = \pi^4$. Therefore, all the fourth-order cumulants of $x_t$ in (12) are zero,
except $k_z(0,0,0) = \pi^4$.

Finally, the sixth order moment of $x_t$ can be derived as:

$$E(x_t - \mu)^6 = E(h_t^6) + 15 \sigma_x^2 E(h_t^4) + 15 \sigma_x^2 E(\xi_t^4) + E(\xi_t^6)$$  \hspace{1cm} (13)

where $E(h_t^6) = 15 \sigma_x^6$. As $\epsilon_t$ is N(0,1), then $E(\xi_t^4) = 10 \mu E(\xi_t^4) + 15 \sigma_x^2 \pi^4 + 10 [E(\xi_t^3)]^2 + 8 \pi^6$, where $\mu = \psi(1/2) - \ln(1/2)$ and $E(\xi_t^3) = -14 \zeta(3)$, where $\psi(\cdot)$ is the Euler psi function and $\zeta(\cdot)$ is the Riemann zeta function; see Abramowitz and Stegun (1970, pp. 258, 807-810, 943).

Therefore, all the moments in (13) are finite, and so is the sixth order moment of $x_t$.

### 3. TESTING FOR SERIAL CORRELATION IN ARSV SERIES

Consider a zero-mean I.I.D. time series, \( \{y_t\}_{t=1,\ldots,T} \), with constant variance. It can be shown that the sample autocorrelations of \( y_t \), given by

$$r_{YY}(k) = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{Y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$  \hspace{1cm} (14)

where $\bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t$, are asymptotically independent and normally distributed with zero mean and variance $1/T$; see, for example, Brockwell and Davies (1991, pp. 222-223).

This result leads to the usual tests for zero autocorrelation in $y_t$, namely, the so-called Barlett 95% confidence bands, $\pm 1.96/\sqrt{T}$, and the Box-Pierce statistic, $Q(K)$, defined as:

$$Q(K) = T \sum_{j=1}^{K} [r_{YY}(j)]^2$$  \hspace{1cm} (15)

Under the null, $H_0: \rho_Y(1) = \ldots = \rho_Y(K) = 0$, $Q(K)$ is asymptotically distributed as a $\chi^2$ with $K$ degrees of freedom.

In models for heteroscedastic time series, the variables $y_t$ are uncorrelated but
dependent, and so the results above do not hold. In this section, we study the properties of $r_Y(k)$ and $Q(K)$ in the basic ARSV process defined in (1).

### 3.1 Asymptotic Properties of the Correlogram of $y_t$

In this subsection we show that the sample autocorrelations of $y_t$ in ARSV processes, are asymptotically normally distributed with zero mean and variance greater than $1/T$. This could lead to errors in the usual tests for uncorrelatedness that are based on these autocorrelations. Appropriate SV-corrected tests are then proposed.

Since the expected value of $y_t$ in ARSV processes is zero, we consider the following sample autocorrelations, $\bar{r}_Y(k) = \bar{c}_Y(k) / \bar{c}_Y(0)$, where $\bar{c}_Y(k) = \frac{1}{T} \sum_{t=1}^{T} y_{t} y_{t-k}$.

**Proposition 1.** Consider the stationary ARSV process in (1) with $|\phi|<1$. The sample autocorrelations $\bar{r}_Y(k)$ are asymptotically independent and normally distributed with zero mean and variance given by

$$C(T,k) = \frac{1}{T} \exp\{\gamma_h(k)\} \quad (16)$$

where $\gamma_h(k)$ is the autocovariance function of $h_t$ given in (5).

The proof is in Appendix A. Notice that the only assumptions on $\varepsilon_t$ needed in the proof are the symmetry and finite fourth order moment of the distribution of $\varepsilon_t$. However, as it is often the case that, in empirical applications it is assumed that $\varepsilon_t$ is Gaussian, in this paper we give the results for this particular case.

**Proposition 1** confirms the result in Taylor (1986, pp.120-122), who proves that the estimate of the variance of the sample autocorrelations that he proposes, converges to expression (16) in series generated by ARSV processes.
If the mean of $y_t$ is unknown and is estimated with its sample mean, the sample autocorrelation $r_Y(k)$, defined in (14), is used instead of $\tilde{r}_Y(k)$. Notice that as $y_t$ is a stationary, uncorrelated zero-mean time series, it can easily be shown, from theorem 1 in Hannan and Heyde (1972), that the sample mean, $\bar{y}$, converges almost sure to the population mean. So the asymptotic behaviour of $r_Y(k)$ and $\tilde{r}_Y(k)$ is the same.

Proposition 1 implies that the usual Barlett 95% confidence bands, $\pm 1.96/\sqrt{T}$, are inappropriate for ARSV processes, and leads directly to the SV-corrected 95% bands given by:

$$\pm 1.96[C(T,k)]^{1/2} = \pm \frac{1.96}{\sqrt{T}} \exp\{0.5\gamma_h(k)\} = \pm \frac{1.96}{\sqrt{T}} \exp\{0.5 \phi^k \sigma^2_{\eta} \frac{1}{1-\phi^2}\}$$  \hspace{1cm} (17)

It is clear that these bands are different for each lag. Moreover, if $\phi>0$, the Barlett bands are “too narrow” and can erroneously reject the hypothesis of uncorrelation. Notice also that the SV-correction factor is bigger, the bigger $\phi$ and $\sigma^2_{\eta}$ and the smaller $k$. Furthermore, note that:

$$\lim_{k \to \infty} C(T,k) = \lim_{k \to \infty} \frac{1}{T} \exp\{\gamma_h(k)\} = \frac{1}{T}$$

since $\gamma_h(k) \to 0$ as $k \to \infty$, and therefore, for large values of $k$, the Barlett confidence bands will be quite close to the SV-corrected bands. As an illustration, figure 1 displays the Barlett and the SV-corrected 95% bands for ARSV processes with $\phi=0.98$ and $\sigma^2_{\eta} = \{0.1, 0.05, 0.01\}$. Clearly, the divergence is different at each lag and the largest divergence occurs at the low-ordered autocorrelations and in the processes with higher variance ($\sigma^2_{\eta}=0.1$). However, the differences become progressively smaller when the lag order
gets larger and/or when the value of $\sigma_\eta^2$ becomes smaller.

The size of the uncorrelatedness test that uses the Barlett 95% bands in ARSV series, may be calculated as follows:

$$p(\text{ERROR I}) = p_{H_0}(\text{r}_Y(k) \geq 1.96/\sqrt{T}) \approx 2 \left[1 - \Phi(1.96 \exp\{-0.5\gamma_\phi(k)\})\right]$$

where $\Phi$ denotes the cumulative distribution function of a standard normal variable. Therefore, if $\phi > 0$, and consequently, $\gamma_\phi(k) > 0$, as it is usual in real data, the probability of type-I error is greater than 0.05. Table 1 shows the values of these probabilities for ARSV processes with $\sigma_\eta^2 = \{0.1, 0.05, 0.01\}$ and $\phi = \{0.5, 0.9, 0.95, 0.98\}$. In this table, it is possible to observe that in ARSV processes with small values of $\phi$, the error with the uncorrected bands is negligible, but in those processes which are closer to the unit root, implying more persistence of the volatility process, and in those processes with higher variance, the error is quite dramatic. In these cases, the correction for volatility is essential to avoid detecting spurious autocorrelations in $y_t$.

In practice, the true parameter values are unknown and the variance of $r_Y(k)$ in (16) must be estimated from the data. As $h_t$ is unobservable, the estimate is constructed from expression (A.4) in Appendix A, as

$$C^*(T,k) = \frac{1}{T} \left[1 + \frac{\hat{\gamma}_Y(k)}{[\hat{\gamma}_Y(0)]^2}\right]$$

(18)

where $\hat{\gamma}_Y(0) = \frac{1}{T} \sum_{i=1}^T y_i^2$ and $\hat{\gamma}_{Y^2}(k) = \frac{1}{T} \sum_{i=k+1}^T (y_i^2 - \hat{\gamma}_Y(0))(y_{i-k}^2 - \hat{\gamma}_Y(0))$. From (18), the following sample SV-corrected 95% bands are obtained:

$$\pm 1.96[C^*(T,k)]^{1/2}$$

(19)
Notice that once we use sample moments to construct the SV-corrected bands in (19), they become the same as those proposed by Diebold (1988) for ARCH processes.

We now turn to the analysis of the Box-Pierce statistic defined in (15). This should also be modified in order to keep the nominal size of the correspondent uncorrelatedness test. From proposition 1, the following correction turns out:

\[
Q_c(K) = \sum_{j=1}^{K} \frac{[r_Y(j)]^2}{C(T, j)} = T \sum_{j=1}^{K} \exp \{-\gamma_h(j)\} [r_Y(j)]^2. \tag{20}
\]

In practice, the asymptotic variance of \( r_Y(k) \) must be estimated from the data and the following sample SV-corrected version of the statistic \( Q(K) \) is obtained:

\[
Q^*_c(K) = \sum_{j=1}^{K} \frac{[r_Y(j)]^2}{C^*(T, j)} = T \sum_{j=1}^{K} \left\{ \frac{[\hat{\gamma}_Y(0)]^2}{[\hat{\gamma}_Y(0)]^2 + \hat{\gamma}_Y^2(j)} \right\} [r_Y(j)]^2. \tag{21}
\]

### 3.2 Finite Sample Properties of the Tests for Uncorrelatedness in \( y_t \)

To analyse the finite sample behaviour of the proposed tests for uncorrelatedness in \( y_t \), we generate series by ARSV processes with parameters \( \phi = \{0.9, 0.95, 0.98\} \), \( \sigma_* = \{0.1, 0.05, 0.01\} \), \( \sigma = 1 \) and \( \epsilon_t \sim \text{NID}(0,1) \). Three sample sizes are considered, \( T=512 \), \( T=1024 \) and \( T=4096 \). For each parameter set and sample size, 5000 independent replicates are generated, and for each replicate, the sample autocorrelations of \( y_t \), up to order 50, are calculated, along with the Box-Pierce statistic in (15) and the SV-corrected Box-Pierce statistics in (20) and (21). All the simulations have been carried out using GAUSS version 3.2 on a Pentium 166 MHz. The Gaussian noise, \( \epsilon_t \), and the AR(1) process, \( h_t \), are generated with commands RNDNS and RECSERAR, respectively.

Table 2 displays the empirical size of the test for uncorrelatedness of order \( k \) in \( y_t \),
for $k=1,10,50$, using the Barlett bands and the SV-corrected bands. The proportion of rejections with the Barlett 95% bands is denoted as $P$, while $P_c$ and $P^*_c$ denote the proportion of rejections with the theoretically SV-corrected and the sample SV-corrected bands in (17) and (19), respectively. In this table, it is possible to observe that, the bigger $\phi$ and/or $\sigma^2$, the bigger the empirical size of the Barlett confidence bands. For instance, when $\{\phi=0.98, \sigma^2=0.05\}$ and $k=10$, the empirical size of the test in a sample of size $T=512$ is more than twice the nominal 5% and it becomes four times bigger when $T=4096$. On the other hand, the empirical size of the sample SV-corrected bands, $P^*_c$, is always quite close to the nominal 5%, even in the more persistent cases and with the smaller sample size. Note also that in the nearly unit root case ($\phi=0.98$), the empirical size of the theoretical SV-corrected bands, $P_c$, is always below the nominal 5%, even with sample size $T=4096$. This could be due to the fact that the process with $\phi=0.98$ is so close to being non-stationary that the convergence to the asymptotic distribution in Proposition 1 is very slow and huge sample sizes are needed to achieve reasonable results. In table 2 we also observe that the problem of detecting spurious autocorrelations with the uncorrected Barlett bands becomes less serious in higher-ordered lags, as predicted by the asymptotic theory. Finally, it is also worth noting that the values of $P$ in table 2 approximately equal the probabilities of type-I error displayed in table 1, obtained using the asymptotic distribution of $r_Y(k)$.

As an illustration, figure 2 represents histograms of the empirical distribution of $\sqrt{T}r_Y(k)$, $r_Y(k)[C(T,k)]^{1/2}$ and $r_Y(k)/[C^*(T,k)]^{1/2}$ together with the standard Normal density, for $k=1,10,50$, in an ARSV process with $\{\phi=0.98, \sigma^2=0.05\}$ and $T=\{512, 4096\}$. This figure shows up clearly that the empirical size of $\sqrt{T}r_Y(k)$ is larger than the nominal
5%, especially in the low-order lags, while the asymptotic distribution seems to provide an adequate approximation to the empirical distribution of \( r_\gamma(k)/[C^*(T,k)]^{1/2} \). On the other hand, it is also possible to observe that the theoretically SV-corrected distribution is not a good approximation in small samples when \( \phi=0.98 \). However, as \( T \) increases the approximation becomes much better.

Table 3 shows, for the same ARSV processes considered in Table 2, the empirical sizes obtained with the Box-Pierce statistic in (15) and the two SV-corrected statistics in (20) and (21) when the null hypothesis is \( H_0: \rho_Y(1)=\ldots=\rho_Y(K)=0 \) for \( K=10 \) and 50. These values, denoted by \( Q, Q_c \) and \( Q^*_c \), respectively, are calculated as the proportion of rejections (out of the 5000 replications) with respect to the 95% percentile of the \( \chi^2_k \) distribution. In this table, it is possible to see that the sample SV-corrected statistic does a good job keeping the nominal size for all the parameter values and lags considered. The largest deviations from nominal size with the Box-Pierce statistic, \( Q(K) \), are always associated with the more persistent processes. Furthermore, in these cases, the SV-corrected statistic \( Q_c(K) \) gives worse results than \( Q^*_c(K) \), especially in the smaller samples. As we said before, this could be due to the fact that the convergence to the asymptotic distribution is very slow when \( \phi \) is close to one.

As an illustration, figure 3 represents histograms of the empirical distribution of the Box-Pierce statistic, \( Q(K) \), and the SV-corrected statistics \( Q_c(K) \) and \( Q^*_c(K) \), together with the \( \chi^2_k \) density, for \( K=10,50 \), in an ARSV process with \( \{\phi=0.98, \sigma_\epsilon^2=0.05\} \) and \( T=4096 \). In this figure, it is possible to observe that the \( \chi^2_k \) distribution is completely inappropriate for \( Q(K) \) and it is not a good approximation to the empirical distribution of \( Q_c(K) \) either. However, the \( \chi^2_k \) distribution fits very well the empirical distribution of the
sample SV-corrected statistic \( Q^x(K) \).

4. SAMPLE AUTOCORRELATIONS OF LOG-SQUARED OBSERVATIONS

As we mentioned before, one of the main tools to identify the presence of stochastic volatility in a given time series is the correlogram of the log-squared observations. In this section, the asymptotic distribution of the sample autocovariances and the sample autocorrelations of the series \( x_t = \log(y_t^2) \) in the ARSV process are analysed. Though the sample autocovariances themselves have a limited use in model identification, its distribution is essential to derive the asymptotic properties of the sample autocorrelations. Finally, the behaviour of the sample autocorrelations in finite samples is also assessed through a Monte Carlo experiment.

Unless otherwise stated, the following definitions of the sample autocovariance and the sample autocorrelation of \( x_t \) will be used:

\[
\begin{align*}
c(k) &= \frac{1}{T} \sum_{t=k+1}^{T} (x_t - \bar{x})(x_{t-k} - \bar{x}) \\
r(k) &= \frac{c(k)}{c(0)} = \frac{\sum_{t=k+1}^{T} (x_t - \bar{x})(x_{t-k} - \bar{x})}{\sum_{t=1}^{T} (x_t - \bar{x})^2}
\end{align*}
\]

where \( \bar{x} \) is the sample mean across all log-squared observations.

4.1 Asymptotic Distribution of the Sample Autocovariances of \( \log(y_t^2) \)

Proposition 2. In the stationary ARSV process defined in (6), the asymptotic distribution of \( \sqrt{T} \{c(0)-\gamma(0),c(1)-\gamma(1),\ldots,c(k)-\gamma(k)\} \) is multivariate Normal with zero mean and variance-covariance matrix \( V \), whose elements, \( \{v_{ij}\}_{i,j=0,\ldots,k} \), are given by:
\[ v_{ij} = \sum_{u=-\infty}^{\infty} [\gamma_h(u)\gamma_h(u+i-j) + \gamma_h(u)\gamma_h(u+i+j)] + 2\sigma_\xi^2 [\gamma_h(0)] + \sigma_\xi^2 I_{\{i=j\}} + (\sigma_\xi^4 + \pi^4) I_{\{i=j=0\}} \]  

(22)

where \(\gamma_h(k)\) is the autocovariance function of the process \(h_t\), \(I_{\{i=j\}}\) is an indicator function defined as \(I_{\{i=j\}}=1\) if \(i=j\), \(I_{\{i=j\}}=0\) if \(i\neq j\), and \(I_{\{i=j=0\}}\) is an indicator function defined similarly.

The proof is in Appendix B.

**Corollary 1.** In the stationary ARSV process defined in (6), the asymptotic marginal distribution of \(\sqrt{T} (c(k)-\gamma(k))\) is Normal with mean zero and variance given by:

\[ v_{kk} = \gamma^2(0) + 2\sigma_h^4 \frac{\phi^2}{1-\phi^2} + \sigma_\xi^2 \phi^{2k} \left[ 2\sigma_\xi^2 + \sigma_h^2 \left( \frac{1+\phi^2}{1-\phi^2} + 2k \right) \right]. \]

This expression follows immediately from (22) considering \(i=j=k\) and replacing \(\gamma_h(u)\) by its value in (5) and working out the summations.

### 4.2 Asymptotic Distribution of the Sample Autocorrelations of \(\log(\gamma_t^2)\)

**Proposition 3.** In the stationary ARSV process defined in (6), the asymptotic distribution of \(\sqrt{T} \{r(1)-\rho(1), \ldots, r(k)-\rho(k)\}'\) is multivariate Normal with zero mean and variance-covariance matrix \(W\) whose elements, \(\{w_{ij}\}_{i,j=1,\ldots,k}\), are given by:

\[ w_{ij} = \sum_{u=-\infty}^{\infty} \rho(u)\rho(u+i-j) + \rho(u)\rho(u+i+j) + 2\rho(i)\rho(j)\rho^2(u) - 2\rho(i)\rho(u+j) - 2\rho(j)\rho(u+j) + \rho(0)\rho(j) \]  

\[ \frac{\gamma^2(0)}{\gamma^2(0)} \]  

(23)

where \(\gamma(0)\) and \(\rho(k)\) are respectively, the variance and autocorrelation function of \(x_t\).

The proof is in Appendix C.

Let us observe the extra term that appears in expression (23) if compared to the usual
expression of the asymptotic covariances of $r(k)$ given by Anderson and Walker (1964) for linear models with I.I.D. disturbances. This extra term accounts for the non-Gaussianity of the disturbance $\xi_t$ in equation (6a). As we mentioned in section 2, this would make the disturbance of the linear representation of $x_t$ in (10) to be non-independent and therefore the conditions in Anderson and Walker (1964) does not hold.

**Corollary 2.** In the stationary ARSV process defined in (6), the sample autocorrelation $r(k)$ is asymptotically normally distributed with mean $\rho(k)$ and variance:

$$\text{Var}(r(k)) = \frac{1}{T} \left\{ 1 + \frac{2[1 + 2\rho^2(k)]}{\gamma^2(0)} \frac{\phi^2}{1 - \phi^2} \sigma^4 + \frac{\rho(2k)}{\gamma(0)} \left[ 2\sigma^2 + \sigma^4 \left( \frac{1 + \phi^2}{1 - \phi^2} + 2k \right) \right] - \frac{4\rho^2(k)}{\gamma(0)} \left[ 2\sigma^2 + \sigma^4 \left( \frac{1 + \phi^2}{1 - \phi^2} + k \right) \right] + \rho^2(k) \left[ 2 + \frac{\pi^4}{\gamma^2(0)} \right] \right\}$$

(24)

The proof of this corollary follows immediately from Proposition 3 noting that:

$$\text{Var}(r(k)) \sim \omega_{wk} \frac{1}{T} \left\{ \sum_{u=-\infty}^{\infty} \rho^2(u) + \rho(u)\rho(u+2k) + 2\rho^2(k)\rho^2(u) - 4\rho(k)\rho(u)\rho(u+k) + \pi^4 \frac{\rho^2(k)}{\gamma^2(0)} \right\}$$

(25)

and replacing in expression (25), $\rho(k)$ by its value in (9) and working out the summations.

An alternative expression of the asymptotic variance of $r(k)$, which will be more useful in practice, can be obtained, after some little effort, from expression (25), as:

$$\text{Var}(r(k)) \sim \frac{1}{T} \left\{ 2\rho(2k) + [1 + 2\rho^2(k)] \left[ 1 + \sum_{u=1}^{\infty} \rho^2(u) \right] - \rho(k) \sum_{u=0}^{k} \rho(u)\rho(k-u) - 8\rho(k) \sum_{u=1}^{\infty} \rho(u)\rho(k+u) + \sum_{u=1}^{2k-1} \rho(u)\rho(2k-u) + 2\sum_{u=1}^{\infty} \rho(u)\rho(u+2k) + \pi^4 \frac{\rho^2(k)}{\gamma^2(0)} \right\}$$

(26)

In practice, the true values of the ACF, $\rho(k)$, are unknown and the variance of $r(k)$ can be
estimated from the data replacing in (26) \( \rho(k) \) by the corresponding sample autocorrelation, \( r(k) \), and truncating the infinite summations at a sufficiently large lag.

Expression (26) can be used to build up confidence bands to test for uncorrelation in the series \( x_t \). Under the null hypothesis \( H_0: \rho(k)=0 \) for any \( k>q \), the confidence bands obtained for the log-squared observations are the same as those obtained in linear models with I.I.D. disturbances; see Brockwell and Davies (1991, pp. 223-224). Though this result seems to be quite unexpected, that is not the case if we observe that the additional term in (25), \( \pi^4 \frac{\rho^2(k)}{\gamma^2(0)} \), vanishes when we compute \( \text{Var}(r(k)) \) under the null \( H_0: \rho(k)=0 \) for any \( k>q \). Therefore, it turns out that, under the null, expression (25) becomes the classical expression of \( \text{Var}(r(k)) \) in Anderson and Walker (1964).

Another important consequence of Proposition 3 is that the sample autocorrelations of the series \( x_t \) in ARSV processes are not asymptotically independent, because the non-diagonal elements of their covariance matrix \( W=\{w_{ij}\} \), defined in (23), are not zero. Therefore, the asymptotic distribution of the Box-Pierce statistic to test for joint uncorrelatedness in the series \( x_t \), would not be a \( \chi^2 \) distribution with \( K \) degrees of freedom. In consequence, using the critical values of such distribution could result in a probability of Type-I error different than the nominal level.

### 4.3 Finite Sample Properties of the Sample Autocorrelations of \( \log(y_t^2) \)

In this subsection we present a simulation study to assess the finite sample properties of the sample autocorrelations of the series \( x_t \). The Monte Carlo design is the same as in section 3.

Table 4 reports the sample bias and standard deviation of \( r(k) \), for \( k=1,10,50 \), together
with the values of the theoretical finite sample bias and the asymptotic standard deviation derived from (24), for some selected cases that are representative of the overall results.

Table 4 also displays the values of the true autocorrelation function $\rho(k)$ of the series $x_t$ calculated with formula (9). The theoretical bias of $r(k)$ for the ARSV process has been derived in Pérez (2000), following the results in Fuller (1996, chap. 6), and is given by the following expression:

$$E(r(k))-\rho(k) = -\frac{\rho(k)}{T} \left( -k + 2 + 4 \sum_{j=1}^{T-1} \left( 1 - \frac{j}{T} \right) \rho^2(j) \right) - \frac{1 - \rho(k)}{T} \left[ 1 + 2 \sum_{j=1}^{T-1} \left( 1 - \frac{j}{T} \right) \rho(j) \right] + \frac{\rho(k)}{T} \left( \pi^4 \gamma^2(0) - \frac{\gamma(0)}{\gamma(0)} \left( 2 + k \right) \right) - \frac{2}{T^2} \left[ 2 \sum_{j=1}^{T-k-1} (T-k-j) \rho(j) \rho(k+j) + (T-k) \sum_{j=0}^{k} \rho(j) \rho(k-j) \right] + O(T^{-2}) \quad (27)$$

Replacing in this equation $\rho(k)$ by its value in (9) and working out the summations, the bias of $r(k)$ can be written, as a function of the parameters of the model, as follows:

$$E(r(k))-\rho(k) = \frac{1}{T} \left[ 1 + \frac{\sigma_h^2}{\gamma(0)} \left( 2 + (k+1) \phi^k \right) \right] + \frac{2 \rho(k)}{T} \left( 1 + \frac{2\phi}{1 - \phi} - k \right) \sigma_h^2 \gamma(0) + \frac{\rho(k)}{T} \left( \pi^4 \gamma^2(0) - \frac{\gamma(0)}{\gamma(0)} \left( 2 + k \right) \right) + \frac{4 \rho(k)}{T} \left( \frac{\sigma_h^4}{\gamma^2(0)} \phi^2 + O(T^{-2}) \right) \quad (28)$$

The main conclusion emerging from the results in Table 4 is that the bias of $r(k)$ is always negative, so the correlogram of $x_t$ always underestimates the corresponding true ACF. However, the bias becomes negligible in large samples. Another important conclusion is that both the bias and variance of $r(k)$ increase with $\phi$ and $\sigma_h^2$. Therefore, the more persistent the volatility and the greater its variance, the worse the sample autocorrelation estimates the true ACF. On the other hand, Table 4 shows that the asymptotic distribution provides an adequate approximation to the finite sample bias and
variance of $r(k)$. However, in the more persistent cases ($\phi=0.98$), the ARSV process is so close to being non-stationary that the asymptotic approximation in proposition 5 is not appropriate in small samples. In fact, the largest differences between the asymptotic and the empirical values always arise in these cases. Finally, from table 4 it is also clear that the usual value $1/\sqrt{T}$ underestimates the standard deviation of the sample autocorrelations of $x_t$, except on those models with $\sigma_\eta^2=0.01$.

As an illustration of the previous results, figure 4 displays the mean correlogram across replications, $\bar{r}(k)$, for $k=1,2,\ldots,50$, together with the corresponding values of the ACF of the series $x_t$, for ARSV processes with $\phi=0.98$. From left to right and top to bottom, the nine correlograms displayed in this figure correspond to $T=512$, $T=1024$ and $T=4096$, and to $\sigma_\eta^2=0.1$, $\sigma_\eta^2=0.05$ and $\sigma_\eta^2=0.01$, respectively. This figure clearly illustrates that there are important biases in the correlogram when the variance is high ($\sigma_\eta^2=0.1$) and $T$ is small ($T=512$). However, the ACF fits quite well the mean correlogram if $\sigma_\eta^2=0.01$ and/or $T=4096$. This sample size is not unusual in financial time series and so in empirical applications with real data, the bias of $r(k)$ would be negligible. Note also that the ARSV process with $\sigma_\eta^2=0.01$ stands for a situation very close to an homocedastic white noise, where the autocorrelations themselves are nearly zero and so is their bias.

To complete the analysis of the sample autocorrelations of the series $x_t$, figure 5 displays the empirical distribution across the 5000 replications of $\sqrt{T} \left[ r(k)-\rho(k) \right]$, $\left[ r(k)-\rho(k) \right]/\left[ V_1(T,k) \right]^{1/2}$ and $\left[ r(k)-\rho(k) \right]/\left[ V_2(T,k) \right]^{1/2}$, for $k=1,10,50$, where $V_1(T,k)$ denotes the asymptotic variance of $r(k)$ in (24), calculated with the true parameter values, and $V_2(T,k)$ denotes the estimated asymptotic variance, computed from expression (26) replacing $\rho(k)$ by its corresponding sample counterpart, $r(k)$. The model considered is an ARSV process.
with parameters $\{\phi=0.98, \sigma^2=0.05\}$ and sample sizes $T=512$ and $T=4096$. The standard Normal density is also drawn over all the empirical distributions in order to check the adequacy of the asymptotic theory developed in section 4.2.

Several important conclusions emerge from figure 5. First, we can see that in ARSV processes, the usual asymptotic variance, $1/T$, is completely inappropriate for sample autocorrelations of the log-squared observations. Secondly, figure 5 confirms that the bias of the sample autocorrelations of the series $x_t$ is quite important when $T=512$. However, if the sample size increases, the asymptotic distribution of the standardized sample autocorrelations using the asymptotic value of $\text{Var}(r(k))$ in (24), becomes a good approximation to the empirical distribution in finite samples. On the other hand, the standard Normal density does not seem to provide an adequate approximation to the standardized distribution with the estimated asymptotic variance.

The approximation to the asymptotic distribution of $r(k)$ in small samples can be further improved by correcting its systematic bias. Figure 6 displays the bias-corrected standardized distribution of $r(k)$, $[r(k)-\rho(k)-\text{bias}(r(k))]/[\text{V}_1(T,k)]^{1/2}$, for $k=1,10,50$, in the ARSV process with $\{\phi=0.98, \sigma^2=0.05\}$ and with sample sizes $T=512$ and $T=4096$. The bias of $r(k)$ has been calculated using formula (28). This figure shows up clearly that the correction for bias is essential when the sample size is small and it is also recommended for larger sample sizes, where the improvement is quite remarkable. In practice, the true parameter values are unknown, and both the bias and the variance of $r(k)$ must be estimated from the data. In this case, the bias of $r(k)$ should be obtained from expression (27) replacing $\rho(k)$ by its corresponding sample counterpart, $r(k)$. However, we have checked that the distribution of $r(k)$ standardized with the estimated bias and variance
does not provide good results in finite samples. In these cases, the bootstrap methodology proposed by Romano and Thombs (1996) would be a good alternative to approximate the empirical distribution of \( r(k) \).

Finally, we carry out some Monte Carlo experiments to assess the finite sample behaviour of the Box-Pierce statistic for the series \( x_t \). Our results show that the \( \chi^2 \) distribution with \( K \) degrees of freedom is not a good approximation to the sample distribution of the Box-Pierce statistic even after correcting the sample autocorrelations by their asymptotic variance. Notice that, as we mentioned before, the sample autocorrelations of \( x_t \) are highly correlated and, therefore, the dependency between the estimated autocorrelations should be taken into account when computing the Box-Pierce statistic. To illustrate the problem, Figure 7 displays several scatter plots with sample autocorrelations of \( x_t \) of order 1, 10 and 50, obtained in the 5000 replicates. These plots show clearly that both sample autocorrelations are highly positively correlated.

The results in this section can explain why, in practice, it is inadequate to use the sample autocorrelations of log-squared observations to identify the model to be used to represent the dynamic evolution of volatility.

5. EMPIRICAL APPLICATION TO THE IBEX-35 INDEX

Figure 8(a) plots daily returns of the IBEX-35 index of the Madrid Stock Market observed from 7/1/87 to 30/12/98 (2991 observations). This series, denoted by \( y_t \), is the result of filtering the original series of returns to remove a small autocorrelation of order one and the effect of two outliers, the mini-crash in Wall Street (13/10/89) and the kidnapping of Gorbachev (19/8/91). The series \( y_t \) moves randomly around a constant zero mean while the volatility evolves over time, with periods of high volatility followed by
periods of low changes. Moreover, the kurtosis of the series is 8.321, indicating a fat tail marginal distribution.

Table 5 displays sample autocorrelations of $y_t$, for some selected lags. Following the results in section 3, table 5 displays the SV-corrected standard deviation of $r_Y(k)$, obtained from expression (18), together with the usual value $1/\sqrt{T}$ for I.I.D. processes. These values are denoted by [s.e.c] and (s.e.), respectively. We can see that some autocorrelations that appear to be significant when compared with the 95% Bartlett bands, become not significant when tested against the corrected SV-bands, $\pm 1.96 [s.e.c]$. This feature shows up even more clearly in Figure 8(b), where the correlogram of $y_t$ up to lag $k=100$ is displayed together with both confidence bands.

Finally, table 6 reports the Box-Pierce statistics for joint uncorrelation of $y_t$ up to order $K$, for $K=5,10,20,50,100$, together with the corresponding SV-corrected Box-Pierce statistic, denoted by $Q(K)$ and $Q_c(K)$, respectively. Once more, we observe that $Q(K)$ is significant for $K=10,20,50,100$. However, when the statistic corrected for volatility, $Q_c(K)$, is used, the hypothesis of uncorrelation in $y_t$ is no longer rejected. Therefore, it is clear that the usual Bartlett confidence bands and the Box-Pierce test can be misleading for processes with time varying volatility.

Regarding the behaviour of the correlations of log-squared returns, table 5 displays the asymptotic standard deviation of the sample autocorrelations of $\log(y_t^2)$ for ARSV processes, together with the usual value $1/\sqrt{T}$ for I.I.D. processes. These values are again denoted by [s.e.c] and (s.e.), respectively. The values of [s.e.c] are obtained from expression (26), by replacing $\rho(k)$ by its corresponding sample autocorrelation, $r(k)$, and truncating the infinite summations at a sufficiently large lag. Figure 8(c) displays the
correlogram of $\log(y_i^2)$ together with the 95% bands obtained recursively under the assumption of conditional homocedasticity as explained in section 4.2. The evidence for conditional heteroscedasticity is overwhelming: the first order autocorrelation of $\log(y_i^2)$ is 0.131 and the Box-Pierce statistic for up to tenth order serial correlation in this series is $Q_{\log}^{10}=609.275$, which is highly significant. It is also possible to observe that the series of log-squared returns have significant autocorrelations even for high lags, with a very slow decay to zero; see Figure 8(c). Notice that the correlogram of $\log(y_i^2)$ may suggest the presence of long-memory in volatility. However, taking into account that the standard deviation of the sample autocorrelations is much bigger than the one used to obtain the confidence bands in Figure 8(c), it is possible that this apparent long memory is spurious.

6. CONCLUSIONS

In this paper we analyse the asymptotic and finite sample properties of the correlogram of series generated by stationary ARSV processes. First, we derive the asymptotic distribution of the correlogram. In particular, we show that the asymptotic variance of the sample autocorrelations is larger than $1/T$. Therefore, the usual tests for uncorrelatedness based on the Barlett bands or the Box-Pierce statistic could lead to detect spurious autocorrelations in series generated by ARSV processes. This effect is especially remarkable in the first lags and in the more persistent cases. Appropriate corrections of both statistics are proposed and it is empirically shown that they provide highly satisfactory results.

We also derive the asymptotic properties of both the sample autocovariances and sample autocorrelations of the series $\log(y_i^2)$. It is shown that the correlogram of this series is always downward biased, and the bias increases with the variance and the
persistence of the volatility. It is also proven that the variance of the sample autocorrelations of \( \log(y_t^2) \) is greater than \( 1/T \). These results can explain why, in practice, it is inappropriate to use the usual confidence bands for the sample autocorrelations of log-squared observations to identify the model adequate to represent the dynamic evolution of the volatility.

The empirical application to the IBEX-35 stock market index illustrates how the usual Barlett bounds and the Box-Pierce statistic could detect spurious autocorrelations in heteroscedastic time series, both in the levels and in the log-squared series.

This paper focused on the basic SV process, where the logarithm of the volatility is an AR(1) process. The results do not extend directly to more general cases. We have some work in progress on the properties of sample autocorrelations of series generated by SV processes with long memory, where the logarithm of the volatility is a fractionally integrated process.

Furthermore, it is possible to use the results on the asymptotic distribution of the sample autocorrelations of log-squared observations to propose a modification of the Box-Pierce statistic that takes into account the correlation between these autocorrelations. We left this correction for further research.

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**APPENDIX A: PROOF OF PROPOSITION 1**

Consider the variable:

$$\sqrt{T} \tilde{\gamma}(k) = \frac{1}{\sqrt{T}} \sum_{t=k+1}^{T} y_t y_{t-k} = \frac{1}{\sqrt{T}} \sum_{t=k+1}^{T} U_t$$

where $U_t = y_t - y_{t-k}$, with $k > 0$. In the stationary ARSV process, it is easily shown that $U_t$ is a martingale difference sequence and therefore, $U_t$ has zero mean and is uncorrelated. Moreover, by the symmetry of $\varepsilon_t$, $E(y_{t_i}^{n_1} \ldots y_{t_r}^{n_r}) = 0$ with $t_i \neq t_j$, if at least one of the exponents $n_i$ is odd (Milhoj 1985). The existence of the fourth moment of $y_t$ is ensured because of the existence of the fourth moment of $\varepsilon_t$. Therefore:

$$\text{Cov}(U_t, U_{t+r}) = E(U_t U_{t+r}) = \begin{cases} 
E(y_t y_{t-k} y_{t+r} y_{t+r-k}) = 0 & \text{if } r \neq 0 \\
E(y_t^2 y_{t-k}^2) = [\gamma_y(0)]^2 + \gamma_{y^2}(k) & \text{if } r = 0 
\end{cases}$$

(A.1)

where $\gamma_y(0)$ is the variance of $y_t$ and $\gamma_{y^2}(k)$ is the autocovariance function of $y_t^2$. The corollary 6.1.1.2 in Fuller (1996) yields the ergodicity of $U_t$. Under these conditions, the central limit theorem for martingales in Billingsley (1961) applies and consequently:

$$\frac{1}{\sqrt{T}} \sum_{t=k+1}^{T} U_t \xrightarrow{d} N(0, E(U_t^2))$$

where $\xrightarrow{d}$ denotes convergence in law. Substituting in this expression $E(U_t^2)$ by its value in (A.1) provides the asymptotic distribution of $\sqrt{T} \tilde{\gamma}(k)$, namely,

$$\sqrt{T} \tilde{\gamma}(k) = \frac{1}{\sqrt{T}} \sum_{t=k+1}^{T} U_t \xrightarrow{d} N(0, [\gamma_y(0)]^2 + \gamma_{y^2}(k))$$
The covariances between the sample autocovariances are calculated by

\[
\text{Cov}(\tilde{c}_y(i), \tilde{c}_y(j)) = \frac{1}{T^2} \sum_{i=1}^{T} \sum_{j=1}^{T} \left[ \text{Cov}(y_1, y_s)\text{Cov}(y_{t-i}, y_{s-j}) + \text{Cov}(y_{t-i}, y_s)\text{Cov}(y_1, y_{s-j}) + \kappa_y(-i, s-t, s-t-j) \right]
\]

where \(\kappa_y(s, r, q)\) is the mixed fourth-order cumulant of \(y_t\). When \(i \neq j, i > 0, j > 0\), the mixed cumulants \(\kappa_y(-i, s-t, s-t-j)\) are zero because all odd moments of \(y_t\) are zero by the symmetry of \(\varepsilon_t\) (Milhoj 1985). Moreover, as the series \(y_t\) is serially uncorrelated, it turns out that \(\text{Cov}(\tilde{c}_y(i), \tilde{c}_y(j)) = 0\) when \(i \neq j, i > 0, j > 0\).

The asymptotic normality of \(\sqrt{T}[\tilde{c}_y(1), \ldots, \tilde{c}_y(k)]\) follows by considering arbitrary linear combinations. Therefore, we have that:

\[
\sqrt{T}[\tilde{c}_y(1), \ldots, \tilde{c}_y(k)] \xrightarrow{d} N\left(0, \begin{bmatrix} [\gamma_Y(0)]^2 + \gamma_Y(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & [\gamma_Y(0)]^2 + \gamma_Y(k) \end{bmatrix} \right)
\]

(A.2)

Finally, from corollary 6.1.1.1 in Fuller (1996) and since the autocovariance function of \(y_t^2\) defined in (4) converges to zero, it is straightforward to prove that:

\[
\tilde{c}_y(0) = \frac{1}{T} \sum_{t=1}^{T} y_t^2 \xrightarrow{p} \gamma_Y(0)
\]

(A.3)

where \(\xrightarrow{p}\) denotes convergence in probability. The conjunction of (A.2) and (A.3) implies that the sample autocorrelations \(\tilde{\tau}_y(k) = \tilde{c}_y(k) / \tilde{c}_y(0)\) are asymptotically independent and jointly normally distributed with a marginal distribution given by

\[
\sqrt{T} \tilde{\tau}_y(k) = \frac{\sqrt{T} \tilde{c}_y(k)}{\tilde{c}_y(0)} \xrightarrow{d} N(0, 1 + \frac{\gamma_{Y^2}(k)}{[\gamma_Y(0)]^2})
\]
Therefore, the asymptotic variance of $\tilde{r}_Y(k)$ can be approximated in finite samples by

$$\text{Var}(\tilde{r}_Y(k)) = \frac{1}{T} \left\{ 1 + \frac{\gamma_{Y^2}(k)}{[\gamma_Y(0)]^2} \right\}$$  \hspace{1cm} (A.4)

Now, substituting in (A.4) $\gamma_Y(0)$ and $\gamma_{Y^2}(k)$ by their values in (2) and (4), respectively, the required expression of $\text{Var}(\tilde{r}_Y(k))$ in (16) is obtained.

**APPENDIX B: PROOF OF PROPOSITION 2**

Consider the vector time series $W_t=(h_t, \xi_t)'$, where $h_t$ and $\xi_t$ are defined as in the linear representation of the ARSV process in (6). The variable $W_t$ can be considered as a linear process such as $W_t=\sum_{j=0}^{\infty} B(j)\nu_{t-j}$, where:

$$W_t=\begin{bmatrix} h_t \\ \xi_t \end{bmatrix}, \nu_t=\begin{bmatrix} \eta_t \\ \xi_t \end{bmatrix}, B(j)=\begin{bmatrix} \beta_{11}(j) & 0 \\ 0 & \beta_{22}(j) \end{bmatrix}$$

The elements $\beta_{11}(j)$, $j=0,1,2,\ldots$, are the coefficients of the Wold representation of the process $h_t$ in (6b), that is, $\beta_{11}(j)=\phi_j$ for every $j$. The coefficients $\beta_{22}(j)$ are all equal to zero, except $\beta_{22}(0)=1$. Moreover, since $\eta_t$ and $\xi_t$ are uncorrelated themselves and independent of each other, the first and second order moments of $\nu_t$ are as follows:

$$E(\nu_t)=\begin{bmatrix} E(\eta_t) \\ E(\xi_t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E(\nu_t\nu_{t}')=\begin{bmatrix} E(\eta_t,\eta_{t'}) & E(\eta_t,\xi_{t'}) \\ E(\xi_t,\eta_{t'}) & E(\xi_t,\xi_{t'}) \end{bmatrix} = \delta(t,s)\begin{bmatrix} \sigma^2_{\eta} & 0 \\ 0 & \sigma^2_{\xi} \end{bmatrix}$$

where $\delta(t,s)$ is the Kronecker delta function, $\delta(t,s)=1$ if $t=s$, $\delta(t,s)=0$ if $t\neq s$.

Let us define the k-th sample autocovariance and cross-covariance of $h_t$ and $\xi_t$ as:
respectively, where \( \bar{h} = \frac{1}{T} \sum_{t=1}^{T} h_{t} \) and \( \bar{\xi} = \frac{1}{T} \sum_{t=1}^{T} \xi_{t} \). Similarly, let \( \gamma_{11}(k) \), \( \gamma_{22}(k) \), \( \gamma_{12}(k) \), \( \gamma_{21}(k) \) be their population counterparts, that is,

\[
\gamma_{11}(k) = \text{Cov}(h_{t}, h_{t+k}) = \gamma_{11}(k), \quad \gamma_{22}(k) = \text{Cov}(\xi_{t}, \xi_{t+k}) = \sigma_{\xi}^{2} I_{\{k=0\}} \tag{B.3}
\]

\[
\gamma_{12}(k) = \text{Cov}(h_{t}, \xi_{t+k}) = 0, \quad \gamma_{21}(k) = \text{Cov}(\xi_{t}, h_{t+k}) = 0 \tag{B.4}
\]

where \( I_{\{k=0\}} \) is the indicator function.

In the stationary ARSV process, the spectra of \( \xi_{t} \) and \( h_{t} \) are both square-integrable. Therefore, the Central Limit Theorem for multivariate linear processes in Hannan (1976) applies to the series \( W_{t} = (h_{t}, \xi_{t})' \). This theorem implies that the \( 4(k+1) \times 1 \) vector:

\[
Z = \sqrt{T} \{ c_{11}(0) - \gamma_{11}(0), \ c_{12}(0), \ c_{21}(0), \ c_{22}(0) - \gamma_{22}(0), \ldots, \ c_{11}(k) - \gamma_{11}(k), \ c_{12}(k), \ c_{21}(k), \ c_{22}(k) \}'
\]

with \( k>0 \), is asymptotically normally distributed with zero mean and a covariance structure given by the following elements:

\[
\lim_{T \to \infty} \text{Cov}(c_{ab}(i) - \gamma_{ab}(i), c_{cd}(j) - \gamma_{cd}(j)) =
\]

\[
= \sum_{u=-\infty}^{\infty} \left[ \gamma_{ac}(u) \gamma_{bd}(u+i-j) + \gamma_{ac}(u) \gamma_{bd}(u+i+j) \right] + \begin{cases} \kappa_{2222} & \text{if } a = b = c = d = 2, i = j = 0 \\ 0 & \text{otherwise} \end{cases}
\]

where \( i=0,1,\ldots,k; \ j=0,1,\ldots,k; \ a=1,2; \ b=1,2; \ c=1,2; \ d=1,2; \) and \( \kappa_{2222} = \text{E}(\xi_{t} \xi_{t}^*) - 3[\text{E}(\xi_{t}^*)]^{2} = \pi^{4} \).

Putting the values of \( \gamma_{11}(k) \), \( \gamma_{22}(k) \), \( \gamma_{12}(k) \) and \( \gamma_{21}(k) \) in (B.3) and (B.4) back into equation (B.5), it is seen that the limiting covariances of \( Z \) are only different from zero when \( \{a=b=c=d=1\}, \ \{a=c=1,b=d=2\}, \ \{a=c=2,b=d=1\}, \ \{a=d=1,b=c=2\}, \ \{a=d=2,b=c=1\} \) and
\{a=b=c=d=2\}. Therefore, the $4(k+1) \times 4(k+1)$ asymptotic variance-covariance matrix of $Z$ can be written as a partitioned matrix as follows:

$$
\Sigma = \begin{bmatrix}
A_{00} & A_{01} & \cdots & A_{0k} \\
A_{k0} & A_{11} & \cdots & A_{1k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{0k} & A_{1k} & \cdots & A_{kk}
\end{bmatrix}
$$

where the $4 \times 4$ matrices $A_{ij}$, for $i,j=0,1,\ldots,k$, are given by:

$$
A_{ij} = \begin{bmatrix}
\sum_{\tau=-\infty}^{\infty} \gamma_h(u)\gamma_h(u+i-j)+\gamma_h(u)\gamma_h(u+i+j) \\
0 & 0 & 0 & 0 \\
0 & 0 & \sigma_{\xi}^2 \gamma_h(i-j) & \sigma_{\xi}^2 \gamma_h(i+j) & 0 \\
0 & \sigma_{\xi}^2 \gamma_h(i+j) & 0 & \sigma_{\xi}^2 \gamma_h(i-j) & 0 \\
0 & 0 & 0 & 0 & \sigma_{\xi}^2 I_{i=j} + (\sigma_{\xi}^2 + \pi^2) I_{i=j=0}
\end{bmatrix}
$$

The asymptotic distribution of the sample autocovariances of $x_t$ follows readily from the distribution of the vector $Z$. Noting that the series $x_t$ in (6a) is the sum of $h_t$ and $\xi_t$, the $k$-th sample autocovariance of $x_t$, say $c(k)$, can be written as:

$$
c(k) = c_{11}(k) + c_{12}(k) + c_{21}(k) + c_{22}(k), \text{ for } k \geq 0 \quad (B.6)
$$

where $c_{11}(k)$, $c_{12}(k)$, $c_{21}(k)$, $c_{22}(k)$ are defined in (B.1) and (B.2). From (7) and (8), the population autocovariance function of $x_t$, $\gamma(k)$, can be decomposed as:

$$
\gamma(k) = \gamma_h(k) + \sigma_{\xi}^2 I_{k=0} = \gamma_{11}(k) + \gamma_{22}(k) \quad (B.7)
$$

From (B.6) and (B.7) it is easily shown that the $(k+1) \times 1$ vector,

$$
U = \sqrt{T} \{ c(0)-\gamma(0), c(1)-\gamma(1), \ldots, c(k)-\gamma(k) \}
$$

is a linear transformation of the vector $Z$ of the form $U = \Omega Z$, where $\Omega$ is the following $(k+1) \times 4(k+1)$ matrix,
As $Z$ is asymptotically normally distributed with zero mean and variance-covariance matrix $\Sigma$, the variable $U = \Omega Z$ converges in law to a multivariate Normal distribution with mean zero and variance-covariance matrix $V = \Omega \Sigma \Omega'$. Working out this product of matrices, it is seen that $V$ is a $(k+1)\times(k+1)$ matrix whose elements $v_{ij}$ are given in (22).

**APPENDIX C: PROOF OF PROPOSITION 3**

Let $c(0)$ and $c(k)$ be the variance and the $k$-th sample autocovariance of the series $x_t$, and $\gamma(0)$ and $\gamma(k)$ their population counterparts. From proposition 2,

$$\sqrt{T} \{c(0) - \gamma(0), c(1) - \gamma(1), \ldots, c(k) - \gamma(k)\}' \overset{d}{\longrightarrow} N(0,V)$$

where the elements of the matrix $V$ are given in (22). Let $g(.)$ be the real vector-valued function from $\mathbb{R}^{k+1}$ into $\mathbb{R}^k$ defined as $g([x_0,x_1,\ldots,x_k]') = (x_1/x_0,\ldots,x_k/x_0)'$, for $x_0 \neq 0$, so that:

$$\sqrt{T} \{r(1) - \rho(1),\ldots,r(k) - \rho(k)\}' = \sqrt{T} \{g([c(0),c(1),\ldots,c(k)]')-g([\gamma(0),\gamma(1),\ldots,\gamma(k)]')\}$$

Then, from the result on the convergence of differentiable functions of asymptotically Normal vectors (Serfling 1980, p. 122),

$$\sqrt{T} \{r(1) - \rho(1),\ldots,r(k) - \rho(k)\}' \overset{d}{\longrightarrow} N(0,VD'V)$$  \hspace{1cm} (C.1)

where $D$ is the matrix of first partial derivatives of $g(.)$ evaluated at $[\gamma(0),\gamma(1),\ldots,\gamma(k)]'$,
Denote the variance-covariance matrix in (C.1) by \( W = D \Sigma D' \) and its \((i,j)\)-th element by \( w_{ij} \). Then, it can be readily shown that \( w_{ij} \) has the following expression:

\[
    w_{ij} = \frac{1}{\gamma^2(0)} \left[ \rho(i)\rho(j)v_{00} - \rho(i)v_{0j} - \rho(j)v_{0i} + v_{ij} \right]
\]  

(C.2)

for \( i,j = 1,2,\ldots,k \). Finally, substituting in (C.2) \( v_{00}, v_{0j}, v_{0i} \) and \( v_{ij} \) by their values in (22), the required expression of \( w_{ij} \) in (23) is obtained.

REFERENCES


Table 1. Asymptotic size of Bartlett 95% confidence bands to test $H_0: \rho_Y(k) = 0$.

<table>
<thead>
<tr>
<th>$\sigma_\eta^2$</th>
<th>Lag</th>
<th>$\phi=0.5$</th>
<th>$\phi=0.9$</th>
<th>$\phi=0.95$</th>
<th>$\phi=0.98$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>k=1</td>
<td>0.058</td>
<td>0.122</td>
<td>0.229</td>
<td>0.570</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>0.050</td>
<td>0.074</td>
<td>0.149</td>
<td>0.485</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
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<td>0.050</td>
<td>0.060</td>
<td>0.216</td>
</tr>
<tr>
<td>0.05</td>
<td>k=1</td>
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<td>0.082</td>
<td>0.125</td>
<td>0.291</td>
</tr>
<tr>
<td></td>
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<td>0.061</td>
<td>0.093</td>
<td>0.242</td>
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<td>k=50</td>
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<td>0.050</td>
<td>0.055</td>
<td>0.119</td>
</tr>
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<td>0.01</td>
<td>k=1</td>
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<td>0.056</td>
<td>0.062</td>
<td>0.083</td>
</tr>
<tr>
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<td>k=10</td>
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<td>0.052</td>
<td>0.057</td>
<td>0.077</td>
</tr>
<tr>
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<td>0.050</td>
<td>0.051</td>
<td>0.061</td>
</tr>
</tbody>
</table>

Table 2. Empirical size of tests for $H_0: \rho_Y(k) = 0$ based on Bartlett bands and on SV-corrected bands.

<table>
<thead>
<tr>
<th>$\sigma_\eta^2$</th>
<th>Lag</th>
<th>Bands</th>
<th>T=512</th>
<th>T=1024</th>
<th>T=4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>k=1</td>
<td>P</td>
<td>0.103</td>
<td>0.165</td>
<td>0.268</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.041</td>
<td>0.030</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.044</td>
<td>0.045</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>P</td>
<td>0.064</td>
<td>0.110</td>
<td>0.197</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.041</td>
<td>0.029</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.045</td>
<td>0.046</td>
<td>0.046</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>P</td>
<td>0.035</td>
<td>0.038</td>
<td>0.058</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.035</td>
<td>0.033</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.036</td>
<td>0.035</td>
<td>0.036</td>
</tr>
<tr>
<td>0.05</td>
<td>k=1</td>
<td>P</td>
<td>0.071</td>
<td>0.102</td>
<td>0.167</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.043</td>
<td>0.039</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.045</td>
<td>0.046</td>
<td>0.048</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>P</td>
<td>0.054</td>
<td>0.079</td>
<td>0.137</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.042</td>
<td>0.038</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.045</td>
<td>0.048</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>P</td>
<td>0.038</td>
<td>0.038</td>
<td>0.053</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.038</td>
<td>0.036</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.038</td>
<td>0.039</td>
<td>0.037</td>
</tr>
<tr>
<td>0.01</td>
<td>k=1</td>
<td>P</td>
<td>0.051</td>
<td>0.054</td>
<td>0.070</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.046</td>
<td>0.045</td>
<td>0.041</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.047</td>
<td>0.047</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>P</td>
<td>0.044</td>
<td>0.053</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
<td>0.043</td>
<td>0.043</td>
<td>0.040</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.045</td>
<td>0.044</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>P</td>
<td>0.040</td>
<td>0.040</td>
<td>0.043</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_c$</td>
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<td>0.040</td>
<td>0.034</td>
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<tr>
<td></td>
<td></td>
<td>$P_c^*$</td>
<td>0.039</td>
<td>0.039</td>
<td>0.040</td>
</tr>
</tbody>
</table>

NOTE: $P$, $P_c$, and $P_c^*$ denote the empirical sizes based on the usual Bartlett bands, on the theoretically SV-corrected bands in (17) and on the sample SV-corrected bands in (19), respectively. The calculations are based on 5000 replications. The nominal size is 5%.
Table 3. Empirical size of Box-Pierce test and SV-corrected Box-Pierce tests for \( H_0: \rho_Y(1)=\ldots=\rho_Y(K)=0 \).

<table>
<thead>
<tr>
<th>( \sigma_n^2 )</th>
<th>Lag Bands</th>
<th>( \phi=0.9 )</th>
<th>( \phi=0.95 )</th>
<th>( \phi=0.98 )</th>
<th>( \phi=0.9 )</th>
<th>( \phi=0.95 )</th>
<th>( \phi=0.98 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 K=10 Q</td>
<td>0.167</td>
<td>0.379</td>
<td>0.661</td>
<td>0.191</td>
<td>0.440</td>
<td>0.798</td>
<td>0.202</td>
</tr>
<tr>
<td>Q_c</td>
<td>0.048</td>
<td>0.032</td>
<td>0.001</td>
<td>0.045</td>
<td>0.045</td>
<td>0.006</td>
<td>0.057</td>
</tr>
<tr>
<td>Q_c^*</td>
<td>0.046</td>
<td>0.048</td>
<td>0.050</td>
<td>0.049</td>
<td>0.050</td>
<td>0.053</td>
<td>0.056</td>
</tr>
<tr>
<td>K=50 Q</td>
<td>0.097</td>
<td>0.277</td>
<td>0.653</td>
<td>0.118</td>
<td>0.309</td>
<td>0.855</td>
<td>0.131</td>
</tr>
<tr>
<td>Q_c</td>
<td>0.042</td>
<td>0.034</td>
<td>0.000</td>
<td>0.047</td>
<td>0.053</td>
<td>0.005</td>
<td>0.048</td>
</tr>
<tr>
<td>Q_c^*</td>
<td>0.043</td>
<td>0.051</td>
<td>0.059</td>
<td>0.045</td>
<td>0.049</td>
<td>0.062</td>
<td>0.044</td>
</tr>
<tr>
<td>0.05 K=10 Q</td>
<td>0.103</td>
<td>0.202</td>
<td>0.437</td>
<td>0.107</td>
<td>0.228</td>
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</tr>
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<td>Q_c</td>
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<td>0.043</td>
<td>0.014</td>
<td>0.046</td>
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<tr>
<td>Q_c^*</td>
<td>0.048</td>
<td>0.051</td>
<td>0.052</td>
<td>0.048</td>
<td>0.049</td>
<td>0.051</td>
<td>0.055</td>
</tr>
<tr>
<td>K=50 Q</td>
<td>0.063</td>
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<td>0.073</td>
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<td>0.647</td>
<td>0.079</td>
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<tr>
<td>Q_c</td>
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<td>0.012</td>
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<td>0.047</td>
<td>0.035</td>
<td>0.048</td>
</tr>
<tr>
<td>Q_c^*</td>
<td>0.041</td>
<td>0.050</td>
<td>0.058</td>
<td>0.045</td>
<td>0.048</td>
<td>0.063</td>
<td>0.046</td>
</tr>
<tr>
<td>0.01 K=10 Q</td>
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<td>0.066</td>
<td>0.109</td>
<td>0.057</td>
<td>0.069</td>
<td>0.129</td>
<td>0.063</td>
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<tr>
<td>Q_c</td>
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<td>0.048</td>
<td>0.039</td>
<td>0.048</td>
<td>0.046</td>
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<td>0.055</td>
</tr>
<tr>
<td>Q_c^*</td>
<td>0.049</td>
<td>0.049</td>
<td>0.051</td>
<td>0.049</td>
<td>0.049</td>
<td>0.049</td>
<td>0.054</td>
</tr>
<tr>
<td>K=50 Q</td>
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<td>0.123</td>
<td>0.052</td>
<td>0.064</td>
<td>0.154</td>
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</tr>
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<td>Q_c</td>
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<tr>
<td>Q_c^*</td>
<td>0.041</td>
<td>0.043</td>
<td>0.050</td>
<td>0.045</td>
<td>0.046</td>
<td>0.049</td>
<td>0.046</td>
</tr>
</tbody>
</table>

NOTE: Q, Q_c and Q_c^* denote the empirical sizes based on the usual Box-Pierce statistic in (15), the SV-corrected Box-Pierce statistic in (20) and the sample SV-corrected Box-Pierce statistic in (21), respectively. The calculations are based on 5000 replications. The nominal size is 5%.
Table 4. Bias and standard deviation (in parenthesis) of sample autocorrelations of $x_t = \log(y_t^2)$, together with autocorrelation function (ACF) values, for $k=1,10,50$. 

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Lag</th>
<th>ACF</th>
<th>T=512</th>
<th>T=1024</th>
<th>T=4096</th>
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<td></td>
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<td>Theoretical</td>
<td>MonteCarlo</td>
<td>Theoretical</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>Lag</td>
<td>ACF</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(Lagrangian)</td>
<td>(Monte Carlo)</td>
</tr>
<tr>
<td>${\phi=0.9, \sigma^2_\eta=0.1}$</td>
<td>k=1</td>
<td>0.087</td>
<td>-0.005</td>
<td>-0.006</td>
<td>-0.003</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.050)</td>
<td>(0.049)</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>0.034</td>
<td>-0.006</td>
<td>-0.007</td>
<td>-0.003</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.047)</td>
<td>(0.046)</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>0.001</td>
<td>-0.005</td>
<td>-0.005</td>
<td>-0.003</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.046)</td>
<td>(0.043)</td>
</tr>
<tr>
<td>${\phi=0.95, \sigma^2_\eta=0.1}$</td>
<td>k=1</td>
<td>0.164</td>
<td>-0.014</td>
<td>-0.015</td>
<td>-0.007</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.062)</td>
<td>(0.060)</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>0.103</td>
<td>-0.017</td>
<td>-0.018</td>
<td>-0.008</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.060)</td>
<td>(0.058)</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>0.013</td>
<td>-0.016</td>
<td>-0.015</td>
<td>-0.008</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.055)</td>
<td>(0.048)</td>
</tr>
<tr>
<td>${\phi=0.98, \sigma^2_\eta=0.1}$</td>
<td>k=1</td>
<td>0.332</td>
<td>-0.060</td>
<td>-0.055</td>
<td>-0.030</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.110)</td>
<td>(0.095)</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>0.277</td>
<td>-0.069</td>
<td>-0.064</td>
<td>-0.034</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.115)</td>
<td>(0.097)</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>0.123</td>
<td>-0.084</td>
<td>-0.073</td>
<td>-0.042</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.117)</td>
<td>(0.080)</td>
</tr>
<tr>
<td>${\phi=0.98, \sigma^2_\eta=0.05}$</td>
<td>k=1</td>
<td>0.200</td>
<td>-0.039</td>
<td>-0.037</td>
<td>-0.019</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.086)</td>
<td>(0.075)</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>0.166</td>
<td>-0.044</td>
<td>-0.041</td>
<td>-0.022</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.087)</td>
<td>(0.074)</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>0.074</td>
<td>-0.050</td>
<td>-0.045</td>
<td>-0.025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.082)</td>
<td>(0.060)</td>
</tr>
<tr>
<td>${\phi=0.98, \sigma^2_\eta=0.01}$</td>
<td>k=1</td>
<td>0.048</td>
<td>-0.011</td>
<td>-0.011</td>
<td>-0.006</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.050)</td>
<td>(0.049)</td>
</tr>
<tr>
<td></td>
<td>k=10</td>
<td>0.040</td>
<td>-0.012</td>
<td>-0.012</td>
<td>-0.006</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.050)</td>
<td>(0.047)</td>
</tr>
<tr>
<td></td>
<td>k=50</td>
<td>0.018</td>
<td>-0.013</td>
<td>-0.013</td>
<td>-0.007</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.048)</td>
<td>(0.043)</td>
</tr>
</tbody>
</table>

NOTE: For each lag and each sample size, the theoretical bias and standard deviation (in parenthesis), calculated from (27) and (24), respectively, are displayed together with their Monte Carlo counterparts, based on 5000 replications.
Table 5. Sample autocorrelations of returns and log-squared returns of IBEX-35.

<table>
<thead>
<tr>
<th>Lag</th>
<th>Series</th>
<th>k=1</th>
<th>k=2</th>
<th>K=3</th>
<th>k=4</th>
<th>k=5</th>
<th>k=10</th>
<th>k=20</th>
<th>k=50</th>
<th>k=100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( y_t )</td>
<td>0.002</td>
<td>0.011</td>
<td>-0.010</td>
<td>0.033</td>
<td>-0.007</td>
<td>0.060_{ab}</td>
<td>-0.049^{a}</td>
<td>-0.018</td>
<td>-0.011</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td></td>
</tr>
<tr>
<td>[s.e.c]</td>
<td>[0.029]</td>
<td>[0.034]</td>
<td>[0.028]</td>
<td>[0.033]</td>
<td>[0.030]</td>
<td>[0.029]</td>
<td>[0.025]</td>
<td>[0.022]</td>
<td>[0.021]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log(( y_t^2 ))</td>
<td>0.131_{ab}</td>
<td>0.183_{ab}</td>
<td>0.172_{ab}</td>
<td>0.107_{ab}</td>
<td>0.142_{ab}</td>
<td>0.129_{ab}</td>
<td>0.118_{ab}</td>
<td>0.091^{a}</td>
<td>0.077^{a}</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td>(0.018)</td>
<td></td>
</tr>
<tr>
<td>[s.e.c]</td>
<td>[0.052]</td>
<td>[0.049]</td>
<td>[0.049]</td>
<td>[0.053]</td>
<td>[0.051]</td>
<td>[0.051]</td>
<td>[0.051]</td>
<td>[0.051]</td>
<td>[0.045]</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: (s.e.) denote the asymptotic standard error of the sample autocorrelations in I.I.D. series, 1/\( \sqrt{T} \). The values [s.e.c] denote the SV-corrected standard error calculated from equations (18) and (26), for the series \( y_t \) and log(\( y_t^2 \)), respectively.

^{a} Significant values with respect to Barlett 95% bounds ±1.96/\( \sqrt{T} \).

^{b} Significant values with respect to the corresponding SV-corrected 95% bounds ±1.96[s.e.c].

Table 6. Box-Pierce statistic of returns and log-squared returns of IBEX-35.

<table>
<thead>
<tr>
<th>Lag</th>
<th>Statistic</th>
<th>K=5</th>
<th>K=10</th>
<th>K=20</th>
<th>K=50</th>
<th>K=100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q(K)</td>
<td>4.024</td>
<td>20.199^{*}</td>
<td>45.559^{*}</td>
<td>82.703^{*}</td>
<td>133.573^{*}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.403)</td>
<td>(0.017)</td>
<td>(0.001)</td>
<td>(0.002)</td>
<td>(0.012)</td>
</tr>
<tr>
<td></td>
<td>Q_{c}(K)</td>
<td>1.276</td>
<td>7.691</td>
<td>20.023</td>
<td>42.765</td>
<td>86.053</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.866)</td>
<td>(0.566)</td>
<td>(0.393)</td>
<td>(0.723)</td>
<td>(0.820)</td>
</tr>
<tr>
<td></td>
<td>Q_{log}(K)</td>
<td>335.677^{*}</td>
<td>609.275^{*}</td>
<td>1043.247^{*}</td>
<td>1936.319^{*}</td>
<td>2830.897^{*}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
</tbody>
</table>

NOTE: Q(K) is the Box-Pierce statistic to test for joint uncorrelation in the series of returns; Q_{c}(K) is the SV-corrected Box-Pierce statistic defined in (21); Q_{log}(K) is the Box-Pierce statistic to test for joint uncorrelation in the series of log-squared returns. The values in parenthesis are the p-values of the corresponding \( \chi^2 \) distribution.

^{*} Significant values at 5% level.
Figure 1. Barlett 95% confidence bands (solid line) and SV-corrected 95% bands for ARSV processes with $\phi=0.98$ and $\sigma_0^2=0.1$ (dashed line), $\sigma_0^2=0.05$ (short-dashed line) and $\sigma_0^2=0.01$ (dotted line).
Figure 2. Empirical distributions (uncorrected, theoretically SV-corrected and empirically SV-corrected) of the sample autocorrelations of $y_t$ for $k=1, 10, 50$, in an ARSV model with $\phi=0.98$, $\sigma^2=0.05$.

(a) $T=512$

(b) $T=4096$
Figure 3. Empirical distributions of Box-Pierce statistic, $Q(K)$, and the two SV-corrected statistics, $Q_c(K)$ and $Q^*_c(K)$, for $K=10, 50$, in an ARSV model with $\phi=0.98$, $\sigma_\eta=0.05$, $T=4096$. 
Figure 4. Mean correlogram (vertical bars) and true autocorrelation function (solid line) of $\log(y_i^2)$ in ARSV processes with $\phi=0.98$, $\sigma^2_\eta=(0.1, 0.05, 0.01)$ and $T=(512, 1024, 4096)$. 

\[ \sigma^2_\eta = 0.1; T = 512 \]

\[ \sigma^2_\eta = 0.05; T = 512 \]

\[ \sigma^2_\eta = 0.01; T = 512 \]

\[ \sigma^2_\eta = 0.1; T = 1024 \]

\[ \sigma^2_\eta = 0.05; T = 1024 \]

\[ \sigma^2_\eta = 0.01; T = 1024 \]

\[ \sigma^2_\eta = 0.1; T = 4096 \]

\[ \sigma^2_\eta = 0.05; T = 4096 \]

\[ \sigma^2_\eta = 0.01; T = 4096 \]
Figure 5. Standardised distributions (uncorrected, theoretically SV-corrected and empirically SV-corrected) of the sample autocorrelations of $x_t = \log(\gamma_t^2)$ for $k=1, 10, 50$, in an ARSV model with $\phi=0.98$, $\eta_2=0.05$.

(a) $T=512$

(b) $T=4096$
Figure 6. Standardised distribution of the sample autocorrelations of $x_t = \log(y_t^2)$ theoretically SV-corrected for bias and variance, for $k=1, 10, 50$, in an ARSV model with $\phi=0.98$, $\sigma_\eta^2=0.05$ and $T\{512,4096\}$.

(a) $T=512$

(b) $T=4096$
Figure 7. Scatter plot of sample autocorrelation coefficients, \( \{r(i) - \rho(i)\} \) versus \( \{r(j) - \rho(j)\} \), of the series \( x_t = \log(y_t) \) of an ARSV model with \( \phi = 0.98 \), \( \sigma_\eta^2 = 0.05 \) and sample sizes \( T = 512 \) (top) and \( T = 4096 \) (bottom).
Figure 8. Daily returns of IBEX-35 from 7/1/87 to 30/12/98 together with the correlograms of returns and log-squared returns.