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CAPITAL STRUCTURE: OPTIMAL LEVERAGE AND MATURITY CHOICE IN A DYNAMIC MODEL

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Abstract

We introduce a dynamic model of optimal capital structure. Analytic solutions for the value of debt and equity are provided when the firm has the opportunity to issue new debt optimally at maturity of current debt. By assuming that this debt consists in a regular coupon bond, and by acknowledging that non positive profits bring the loss of tax deductions, but do not necessarily lead to corporate default, we solve some of the limitations in Kane, Marcus and McDonald (1985) zero coupon bond model, and in Fischer, Heinkel and Zechner (1989) perpetual debt model. A numerical algorithm is proposed to solve the optimization problem. Simulation results indicate that the model is able to replicate standard leverage ratios, debt maturities and credit spreads for reasonable parameter values.

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1 Introduction

The capital structure of a firm is basically described by the two main elements that characterize its debt: Leverage and maturity. Understanding the trade-offs that determine the optimal choice of these two variables, is therefore a prerequisite to develop a model able to provide quantitative guidance.

Optimal leverage represents a compromise between the “nominal” stream of tax benefits that debt generates, and the probability of this stream being received. As the firm increases its leverage, interest payments are higher, and also higher the tax deductions associated to these payments. This would induce the firm to choose a leverage rate equal to 100 percent. However, there are two reasons why a firm will never do this: First, higher leverage means higher default probability, with the costs associated to this event. Second, and more importantly, tax deductions only apply if the firm has a taxable income, that is, if the firm is getting positive profits, and these are less likely to happen the higher the interest payments.

On the other hand, the optimal debt maturity choice will balance “firm flexibility” and issuance costs. The shorter the maturity, the higher the firm flexibility to accommodate its leverage to new information.[1] But at the same time, the shorter the maturity, the more often the firm has to face issuance costs. The trade-off between these two contrary incentives will determine the optimal debt maturity.

One approach to the problem of the optimal choice of the capital structure, has been to assume that by levering the firm managers can increase its market value. The measure of the tax advantage to debt is then the difference between the market value of an optimally levered firm, and that of its unlevered counterpart. This approach has been widely adopted in recent years, and includes Brennan and Schwartz (1978), Leland (1994), Leland and Toft (1996), Leland (1998), and many others.

A different approach to the problem, and the one adopted here, was introduced by Kane, Marcus and McDonald (1985). In their own words[2]:

“Equilibrium in the market for real assets requires that the price

[1] Although this new information may include fluctuations in the risk free interest rate, the firm per period income, and may other innovations, we will assume that the only source of uncertainty is the market value of the firm unlevered assets.

[2] Kane, Marcus and McDonald (1985, p. 479).

> of those assets be bid up to reflect the tax shields they can offer to levered firms. Thus, there must be an equality between the market price of real assets and the values of optimally levered firms. The standard measure of the advantage to leverage compares the values of levered and unlevered assets, and can be misleading and difficult to interpret..."

Kane, Marcus and McDonald (KMM) show that a more reasonable measure of the advantage to leverage is the "extra rate of return, net of a market premium for bankruptcy risk, earned by a levered firm relative to an otherwise-identical unlevered firm". They find closed form solutions for the value of the debt and the equity, and develop a numerical algorithm to find the optimal leverage and maturity for different parameter values. By identifying the value of an optimally levered firm with that of its unlevered counterpart, they recognize the possibility of optimally levering the firm after the maturity date of current debt.

Although the trade-offs described above are also the basic hypothesis in KMM (1985), and although we acknowledge in their article the motivation for the present one, we have to claim that their formulation of the problem results in an inadequate "theoretical test" for such hypothesis.

We have argued that the main element that will prevent the firm from choosing a 100 percent leverage rate, is the possibility of getting non positive profits, because tax benefits are lost in this case. KMM, however, link the loss of tax benefits to the situation in which the firm goes to bankruptcy, which is a much more extreme (and unlikely) situation. Being this point important, the main limitation of their model comes nevertheless from the assumption of a zero coupon bond. Brennan and Schwartz (1978) introduced a model in which the firm issued a coupon bond, was levered during the holding period, and then remained permanently unlevered. This model predicted that perpetual debt was the optimal choice. KMM then argued that perpetual debt has to be the optimal choice if the firm can issue debt only once, because under this assumption this is the only way in which the firm can enjoy tax benefits indefinitely. A model that allows the firm to issue new debt optimally after the maturity of current debt, should not give such a questionable result. The argument, being appealing, cannot however be "tested" using a zero coupon bond model. The reason is that a model of this type will result in finite optimal maturity, even if the firm cannot issue new debt at expiration. We show this by an example in Appendix 1, but we provide the basic intuition here: With a zero coupon bond interests are paid at the end of the holding period. Because no coupon is paid during this period, interests accumulate exponentially. For short maturities, higher interests and the corresponding tax deductions more than offset the increment in the default probability that follows from a much higher final payment and a longer maturity. A higher default probability has two effects: First, it reduces the present value of tax benefits because these are enjoyed if and only if the firm does not default (following Brennan and Schwartz (1978) we assume in this case that tax deductions are not directly linked to the presence of positive profits). Second, it increases the present value of bankruptcy costs. Eventually, the increment in nominal tax benefits does not compensate the reduction in the probability of receiving this flow on one hand, nor the increment in the present value of bankruptcy costs on the other hand, and the result is that an optimal maturity has been reached.[3]

Fisher, Heinkel and Zechner (1989) follow the argument by which the value of an optimally levered firm and that of an unlevered one has to be identical in equilibrium. The objective of their model is to analyze how recapitalization costs make firms to deviate from their optimal leverage, and how this can make similar firms to exhibit quite different leverage ratios. Their model considers a quite interesting problem that we acknowledge to avoid to keep our own model tractable, but they do not allow tax benefits to be lost in case of non positive profits, and debt is assumed by them to be perpetual. These two aspects make their model unable to reflect how leverage and maturity are determined according to the trade-offs discussed at the beginning of this section.

We introduce a model in which risk free interest rate, firm risk, bankruptcy costs, issuance costs, tax benefits on debt, and earnings ratio, determine the optimal choice of leverage and maturity. The model assumes that debt pays a regular flow of interests, allows the firm to rebalance its optimal capital structure at maturity, and considers default to be an endogenous decision: At every period, equity holders decide whether or not they are willing to finance interest payments (or interest payments plus principal at maturity). Unlike previous models, this decision is assumed to be time dependent. Simulation results are also provided, with standard leverage ratios, debt maturities, and credit spreads, being replicated for reasonable parameter values.

[3]There is of course an even more intuitive explanation for this result: It would be difficult to find an investor willing to buy perpetual debt which pays no coupon!.

The rest of the article is organized as follows: Section II presents the model; Formal proofs are left to the appendix in section VI. Section III gives intuition on results found in Section II by considering a simple example based on the well known binomial model. Section IV develops the numerical algorithm that solves the optimization problem, and provides simulation results. Finally section V offers some concluding remarks.

2 The Model

We assume that the value of the firm unlevered assets evolves according to the diffusion process

$$dA = \mu A dt + \sigma A dz \tag{1}$$

where μ is the expected rate of return on A, σ is the volatility of the rate of return which will be assumed to be constant, and dz is the increment of a Wiener process.

Consider the firm issues a coupon bond with principal P and maturity in T years. To keep the analysis as simple as possible, we restrict T to be a natural number, and coupon payments, that we denote by c, to be concentrated at the end of every single year. Debt issuance generates a cost of βP which is borne by equity holders, with $0 < \beta < 1$. All coupon payments, and the final payment of coupon plus principal, are to be financed by issuing additional equity. Tax deductions, τ, on the payment of interests are also presumed. We also assume that these deductions apply as long as the firm is profitable. If the firm annual EBIT represents a constat ratio, ε, of the firm assets value, then this assumption implies that tax deductions will be available in a given period if and only if $\varepsilon A > c$, that is, if and only if $A > Ad$, where $Ad = \frac{c}{\varepsilon}$. Finally, if the firm enters into bankruptcy, its assets A lose a fraction $0 \leq \alpha \leq 1$ of its value due to bankruptcy costs.[4]

[4]Although default need not lead to bankruptcy, we will use interchangeably the terms default and bankruptcy to describe the situation in which the firm defaults. This may in fact lead to any situation from an informal or formal restructuring to a formal liquidation. What happens when a firm defaults is not modeled here.

Note also that the presence of bankruptcy costs makes (1) not to hold at the time the firm defaults.

We denote $S(A,t)$, $D(A,t)$ and $v(A,t)$ the equity, debt and firm value respectively, when t periods remain to maturity. Given that A represents the market value of the firm assets, it must reflect all possible future revenues coming from that assets, including tax benefits net of default costs when the firm is optimally levered. As a consequence, we may identify the market value of an unlevered firm with that of an optimally levered one. However, as long as the firm is unlevered, it losses the return coming from tax benefits net of default costs. In other words, the unlevered firm earns a below-equilibrium rate of return. Following McDonald and Siegel (1984), and KMM (1985), we have that under the assumptions made in Merton's (1973) intertemporal CAPM, any contingent claim F, which underlying asset is A, must satisfy the partial differential equation

$$\frac{1}{2}\sigma^2 A^2 F_{AA} + (r-\delta) A F_A - F_t - rF = 0$$

where δ represents precisely the difference between the equilibrium rate of return μ^* (that necessary to compensate investors for bearing the risk of asset A) and the actual rate of return μ, that is, it represents the difference in return between an unlevered firm and an optimally levered firm. As in KMM (1985), we consider δ to be the appropriate metric of the tax benefits of debt net of default costs. δ will affect the valuation of $S(A,t)$, $D(A,t)$ and $v(A,t)$ in the same way it would do the presence of a dividend yield.

The firm value will be the sum of the equity value and the debt value. However, we also may see the firm value as the sum of two different assets. Specifically

$$v(A,t) = V(A,t) + TB(A,t)$$

$$t = 0, 1, ..., T-1, T^+$$

where $TB(A,t)$ represents the present value of the tax benefits of debt. $V(A,t)$ on the other hand, describes an asset that gives the assets of the firm whenever it becomes unlevered, either because it defaults and goes to debt holders hands, or because the debt finally matures. It will not coincide with the current value of the firm assets for two reasons: First, δ will affect the present value of the firm assets, in the same way the value of a forward contract (with zero delivery price) is affected when the underlying asset pays a dividend yield. Second, if the firm becomes unlevered because it defaults, then the value of those assets will lose a rate α due to bankruptcy costs.[5] Note that we define $v(A,t)$ as the sum of these two components from $t=0$ to $t=T^{+}$. This is to differentiate the firm value after issuance, $v^{+}(A,T)$, from the firm value before issuance, $v^{-}(A,T)$. In this last case, another component must me subtracted: The issuance costs βP. Then

$$v^{-}(A,T) = v^{+}(A,T) - \beta P \tag{2}$$

Assuming by the moment that the bankruptcy-triggering firm assets value when t periods remain to maturity, Ab_t, is exogenous and strictly lower than Ad, it is shown in Appendix 2 that

$$\begin{aligned} S^{-}(A,T) &= S^{+}(A,T) - \beta P \\ &= Ae^{-\delta T} N_T(\mathbf{a}_{T,0}) - Pe^{-rT} N_T(\mathbf{b}_{T,0}) \\ &\quad - \sum_{k=0}^{T-1} ce^{-r(T-k)} \left[N_{T-k}(\mathbf{b}_{T,k}) - \tau N_{T-k}(\mathbf{c}_{T,k})\right] - \beta P \end{aligned} \tag{3}$$

$$\begin{aligned} V(A,T) &= Ae^{-\delta T} N_T(\mathbf{a}_{T,0}) \\ &\quad + (1-\alpha) A \sum_{k=0}^{T-1} e^{-\delta(T-k)} \left[N_{T-(k+1)}(\mathbf{a}_{T,k+1}) - N_{T-k}(\mathbf{a}_{T,k})\right] \end{aligned} \tag{4}$$

$$TB(A,T) = \sum_{k=0}^{T-1} \tau c e^{-r(T-k)} N_{T-k}(\mathbf{c}_{T,k}) \tag{5}$$

$$v^{-}(A,T) = V(A,T) + TB(A,T) - \beta P \tag{6}$$

[5]It would be possible to consider two different assets to represent these two effects. However, the mathematical exposition is simplified by considering both together.

$$
\begin{aligned}
D(A,T) &= v^{-}(A,T) - S^{-}(A,T) = v^{+}(A,T) - S^{+}(A,T) \\
&= (1-\alpha)A\sum_{k=0}^{T-1} e^{-\delta(T-k)}\left[N_{T-(k+1)}(\mathbf{a}_{T,k+1}) - N_{T-k}(\mathbf{a}_{T,k})\right] \\
&\quad +Pe^{-rT}N_T(\mathbf{b}_{T,0}) + \sum_{k=0}^{T-1} ce^{-r(T-k)}N_{T-k}(\mathbf{b}_{T,k}) \qquad (7)
\end{aligned}
$$

where $N_0(\cdot) = 1$. For the more general case of $z \geq 1$, $N_z(\cdot)$ denotes the multivariate normal cumulative distribution function of dimension z, with marginal distribution for each component $N_1(0,1)$, correlation matrix $R_z = \left\{\rho_{ij}^z\right\}$, with[6]

$$
\rho_{ij}^z = \begin{cases} \sqrt{\frac{z-i+1}{z-j+1}} & if \quad i \geq j \\ \sqrt{\frac{z-j+1}{z-i+1}} & if \quad i < j \end{cases}
$$

and integration limits[7]

$$
\mathbf{a}_{t,k} = [a_{t,k}\ a_{t,k+1}\ ...\ a_{t,t-1}]
$$

$$
\mathbf{b}_{t,k} = [b_{t,k}\ b_{t,k+1}\ ...\ b_{t,t-1}]
$$

$$
\mathbf{c}_{t,k} = [c_{t,k}\ b_{t,k+1}\ ...\ b_{t,t-1}]
$$

$$
t = 1, ..., T
$$

$$
k = 0, ..., t-1
$$

[6]Although this extension of KMM (1985) to a coupon bond, is based on the methodology developed by Geske (1977) to value corporate liabilities as compound options, the correlation matrix we provide apparently differs from that derived by Geske. This is a simple question of which is considered to be the "first", "second",..., and "last" variable into the multivariate normal density function. As an example, the element ρ_{12}^z in Geske (1977) would be $\sqrt{\frac{1}{2}}$, while this is the value of the element $\rho_{(z-1)z}^z$ in our case.

[7]$a_{t,k}$ is the integration limit of the "first" variable in the multivariate normal density function, while $a_{t,t-1}$ that of the "last" one.

where

$$a_{t,s} = \frac{\ln\left(\frac{A_t}{Ab_s}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}$$

$$b_{t,s} = a_{t,s} - \sigma\sqrt{t-s}$$

$$c_{t,s} = \frac{\ln\left(\frac{A_t}{Ad}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}$$

$$s = k, ..., t-1$$

with A_t denoting the firm assets value when t periods remain to maturity

We may now derive the endogenous bankruptcy-triggering threshold, Ab_t, as the A value below which equity holders choose not to pay when t periods remain to maturity. It is reasonable to assume that the firm assets value that makes the firm to default, is lower than the one that makes it to lose tax deductions.[8] The consequence is that we may presume that the payment that equity holders choose not to satisfy bringing default is $P + c$ and not $P + c(1-\tau)$ at maturity, and c and not $c(1-\tau)$ at any other period. In order to derive the T critical threshold values we should proceed recursively, that is, we can find first Ab_0 as the solution to

$$S(A, 0) = A - P - c = 0$$

getting, $Ab_0 = P + c$, and then find $Ab_1, Ab_2, ..., Ab_{T-1}$ sequentially as the implicit solution to

$$\begin{aligned} S(A,t) &= Ae^{-\delta t}N_t(\mathbf{a}_{t,0}) - Pe^{-rt}N_t(\mathbf{b}_{t,0}) \\ &\quad - \sum_{k=0}^{t-1} ce^{-r(t-k)}\left[N_{T-k}(\mathbf{b}_{T,k}) - \tau N_{T-k}(\mathbf{c}_{T,k})\right] - c = 0 \end{aligned} \tag{8}$$

for $t = 1, ..., T-1$.

[8] Typically, default will be preceded by a long period of continuous negative profits in which tax deductions are not enjoyed.

3 An Example: The Binomial Case for $T = 2$

In order to give some intuition about what is behind previous expressions, we provide a simple example based on the well known binomial model for $T = 2$.[9] In the general case we have presented above, a total of three "states of nature" are feasible at each "node", namely, non default and enjoy tax deductions, non default but lose tax deductions, and default. To proceed with such an example we have then to reduce the set of feasible states of natures, and this is done by assuming that tax deductions are obtained in any non defaulting state, that is, Ad is not constant in this case but equal to Ab_t when t periods remain to maturity.

Picture 1 describes the possible evolution in the market value of the underling asset A, and therefore, that of an optimally levered firm v.

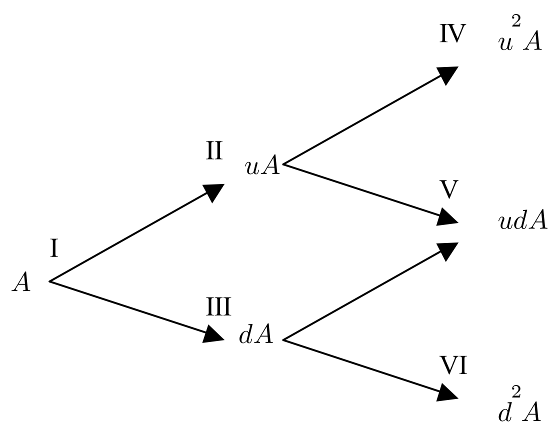

Figure 1: Possible evolution of A.

In a risk neutral world, any agent should expect to get the risk free interest rate as compensation for holding an optimally levered firm. Then at II

$$uv = e^{-r} \left[pu^2v + (1-p)\, udv + pu^2v \left(e^{\delta} - 1\right) + (1-p)\, udv \left(e^{\delta} - 1\right)\right]$$

where p denotes the risk neutral probability of an upward movement u, and the last two terms follow from the fact that, in addition to capital gains

[9] The author thanks J. Ignacio Peña for proposing him to include this example.

(or losses), we have to consider the revenues coming from an optimal leverage. As a result

$$p = \frac{e^{(r-\delta)} - d}{u - d}$$

It is easily shown that the same p applies at III and I. We can now use this risk neutral probability to evaluate the equity value at I, S_I, recursively:

At IV

$$S_{IV} = Max\left\{0, u^2 A - P - (1-\tau)\, c\right\}$$

Whether or not S_{IV} is strictly greater than zero will depend on the specific parameters. We assume $u^2 A - P - (1-\tau)\, c > 0$. In which follows, $=_a$will denote that the statement $=$ holds by assumption. Then

$$S_{IV} =_a u^2 A - P - (1-\tau)\, c$$

At V

$$\begin{aligned} S_V &= Max\left\{0, udA - P - (1-\tau)\, c\right\} \\ &=_a udA - P - (1-\tau)\, c \end{aligned}$$

and at VI

$$\begin{aligned} S_{VI} &= Max\left\{0, d^2 A - P - (1-\tau)\, c\right\} \\ &=_a 0 \end{aligned}$$

At II

$$
\begin{aligned}
S_{II} &= Max\left\{0, e^{-r}\left[u^2A - P - (1-\tau)\,c\right]p\right. \\
&\qquad \left. + e^{-r}\left[udA - P - (1-\tau)\,c\right](1-p) - (1-\tau)\,c\right\} \\
&= {}_ae^{-r}\left[u^2Ap + udA\,(1-p)\right] - e^{-r}\left[P + (1-\tau)\,c\right] - (1-\tau)\,c
\end{aligned}
$$

At III

$$
\begin{aligned}
S_{III} &= Max\left\{0, e^{-r}\left[udA - P - (1-\tau)\,c\right]p - (1-\tau)\,c\right\} \\
&= {}_ae^{-r}udAp - e^{-r}\left[P + (1-\tau)\,c\right]p - (1-\tau)\,c
\end{aligned}
$$

Finally, at I$^-$

$$
\begin{aligned}
S_I^- &= \left[u^2Ap^2 + udA2p\,(1-p)\right]e^{-2r} - Pe^{-2r}p\,(2-p) \\
&\qquad - (1-\tau)\,ce^{-2r}p\,(2-p) - (1-\tau)\,ce^{-r} - \beta P
\end{aligned} \tag{9}
$$

If we know set $T = 2$ in (3), and Ad equal to Ab_t whenever t periods remain to maturity, then

$$
\begin{aligned}
S^-(A,2) &= Ae^{-2\delta}N_2\left(\mathbf{a}_{2,0}\right) - Pe^{-2r}N_2\left(\mathbf{b}_{2,0}\right) \qquad (10) \\
&\qquad - (1-\tau)\,ce^{-2r}N_2\left(\mathbf{b}_{2,0}\right) - (1-\tau)\,ce^{-r}N_1\left(\mathbf{b}_{1,0}\right) - \beta P
\end{aligned}
$$

$Ae^{-2\delta}N_2(\mathbf{a}_{2,0})$ then represents the present value of the firm unlevered assets contingent on non defaulting at any period. In (9) this translates into the value of the firm assets in all possible non defaulting final nodes, times the risk neutral probability of those nodes being reached, and discounted at the risk free interest rate. $Pe^{-2r}N_2(\mathbf{b}_{2,0})$ on other hand, represents the present value of the payment of debt principal, where $N_2(\mathbf{b}_{2,0})$ is the risk neutral probability of this payment taking place, that is, $N_2(\mathbf{b}_{2,0})$ is the risk neutral probability of non defaulting at any period, in our example $p^2 + p(1-p) + (1-p)p = p(2-p)$. $(1-\tau)ce^{-2r}N_2(\mathbf{b}_{2,0})$ can be identically interpreted; This is the present value of the final coupon payment. $(1-\tau)ce^{-r}N_1(\mathbf{b}_{1,0})$ represents the present value of the coupon payment to be satisfied when one year remains to maturity. $N_1(\mathbf{b}_{1,0})$ is the risk neutral probability of non defaulting at this period. In our binomial example, we have assumed that in any case (II or III), the firm does not default, and this makes this probability to add up to one. Finally βP indicates the issuance costs shared by equity holders.

It is possible to derive V_I, TB_I, v_I^- and D_I using similar arguments:

$$V_I = \left[u^2Ap^2 + udA2p(1-p)\right]e^{-2r} + (1-\alpha)d^2Ae^{-2r}(1-p)^2 \tag{11}$$

$$TB_I = \tau ce^{-2r}p(2-p) + \tau ce^{-r} \tag{12}$$

$$v_I^- = V_I + TB_I - \beta P \tag{13}$$

$$\begin{aligned} D_I &= v_I^- - S_I^- \\ &= (1-\alpha)d^2Ae^{-2r}(1-p)^2 \\ &\quad + Pe^{-2r}p(2-p) + ce^{-2r}p(2-p) + ce^{-r} \end{aligned} \tag{14}$$

We may relate now D_I with its equivalent in the general case $D(A,2)$[10]

$$\begin{aligned} D(A,2) &= (1-\alpha) A \sum_{k=0}^{1} e^{-\delta(2-k)} \left[N_{2-(k+1)}\left(\mathbf{a}_{T,k+1}\right) - N_{2-k}\left(\mathbf{a}_{2,k}\right) \right] \\ & \quad + P e^{-2r} N_2\left(\mathbf{b}_{2,0}\right) + \sum_{k=0}^{1} c e^{-r(2-k)} N_{2-k}\left(\mathbf{b}_{2,k}\right) \qquad (15) \end{aligned}$$

In any defaulting state, the debt holders get the firm assets net of default costs. This is contained in the first element of (14) and (15). Remember that default only takes place in our example at VI. As a result, this component of the debt value translates into the value of the firm assets at VI, $(1-\alpha) d^2 A$, times the discount factor, e^{-2r}, times the risk neutral probability of this state being reached, $(1-p)^2$. If the firm does not default in any state, then at the final period the debt holders get back the debt principal. This is what the second term in both cases reflect. Finally, these two expressions take into account the present value of future coupon payments.

4 Optimal Capital Structure

The capital structure should be chosen as to maximize the market value of the firm, as a function of the market value of the firm assets. Given that we assume that the market values the firm assets fully reflecting the possibility of an optimal leverage, the best the firm can do is confirm the market expectations. Said in other words, if the firm is unlevered, its market value should reflect the implicit option to choose an optimal leverage. If the firm is suboptimally levered later on, its value will fall reflecting that it has been constrained to have a suboptimal leverage for a given period of time. This is reasonable: Even if the suboptimally levered firm generates tax revenues that the unlevered firm does not produce, the market will give a higher value to the unlevered firm because it maintains the potential for an immediate optimal leverage. As a result, the solution to the optimization problem[11]

[10] Short-hand notation used.
[11] The restriction states that debt should be issued at par.

$$
\begin{aligned}
&Max_{P,T} \quad v^{-}(A,P,T) \\
&s.t\ D(A,P,T) = P
\end{aligned}
\tag{16}
$$

should be that

$$
v^{-}(A,P^{*},T^{*}) = A \tag{17}
$$

But maximizing the firm value is equivalent to maximize the return coming from the tax advantage to debt, which is measured by δ. Therefore, no pare (P,T) that solves (16) and results in (17) for a given δ can be optimal, if there is another pare (P^{*},T^{*}) that solves (16) and results in (17) for a higher δ. The solution will be then given by a vector (P^{*},T^{*},δ^{*}) such that δ^{*} is the highest possible δ for which the solution to (16) is a pare (P^{*},T^{*}), that in addition makes condition (17) to hold. We then propose the following numerical algorithm:[12]

1. Set $A = 100$, and some initial T_0 and δ_0.[13]

2. Search the P value that maximizes v^{-}, given T_0 and δ_0. This requires the following procedure: For any guess of P, search the c value that makes $D = P$. Any guess of c as the solution to $D = P$, implies at the same time to compute $Ab_0, Ab_1, ..., Ab_{T-1}$ sequentially as described in section II.

3. Once the optimal P has been found, check if $v^{-} = A$.

4. If $v^{-} \neq A$, find a new δ_1 such that $v^{-} = A$, given P. Again, any guess of δ_1, implies to search for the c value that makes $D = P$, and any guess of c, implies the computation of $Ab_0, Ab_1, ..., Ab_{T-1}$.

5. Using this new δ_1 instead of δ_0, go back to step 2 and repeat until convergence, that is, until the resulting v^{-} equals A in step 3.

[12] Although this is similar in spirit to that in KMM (1985), there are two main differences: First, we have to compute T bankruptcy-triggering firm assets values that are not present in their model. Second, we force debt to be issued at par, what they do not need to do given their different formulation of the problem.

[13] A can be arbitrarilly fixed given that the model does not include any scale effects.

6. Consider different T values and search for the one that generates the maximum δ.

The resulting vector (P^*, T^*, δ^*) is simultaneously consistent with the value maximizing criterium (steps 2 and 6), and with conditions $v^- = A$ (step 3) and $D = P$ (step 2). Note that the algorithm requires to evaluate multivariate normal cumulative distribution functions of order equal and lower than T. We can approximate these estimations by using Monte Carlo simulations. As an example, consider we need to evaluate $N_2(\mathbf{a}_{2,0})$ for some given $a_{2,0}$ and $a_{2,1}$. In this case we generate 100.000 observations from a bivariate normal density function, with marginal distribution for each component $N_1(0,1)$, and correlation matrix R_2. The result are 100.000 pairs (ξ_1, ξ_2). We then compute the number of times it happens simultaneously that $\xi_1 < a_{2,0}$ *and* $\xi_2 < a_{2,1}$. The ratio of this number of *favorable cases* over the total number of *possible cases*, gives us finally an approximation of $N_2(\mathbf{a}_{2,0})$.[14]

Base case parameters are chosen as follows:

$$\begin{aligned}
r &= 0.04 \\
\sigma &= 0.25 \\
\alpha &= 0.15 \\
\beta &= 0.01 \\
\tau &= 0.25 \\
\varepsilon &= 0.035
\end{aligned}$$

Ibbotson Associates (1997) reports an historical interest rate on U.S. Treasury Bills around 0.037. The standard deviation of the value of unlevered assets is the same used by KMM (1985), and by Fisher, Heinkel and Zechner (1989), and similar to the one applied in other models. Bankruptcy costs are the mean of the range found by Andrade and Kaplan (1998) who estimate financial distress costs to be 10 to 20 percent of firm value. Issuance costs are consistent with estimations provided by Blackwell and Kidwell (1988). They find flotation costs to represent 1.165 percent of the issue size for public issues and 0.795 percent for private issues. Tax advantage to debt is chosen to

[14] The author thanks Santiago Velilla for the suggestion of this method.

	Optimal Lev.	Coupon	Credit Spread	Optimal Mat.	Tax Adv.	Ad
Base Case	43.10	1.99	51.94	6	25.80	56.93
$r = 0.06$	37.94	2.42	17.65	4	34.33	69.06
$\sigma = 0.1$	68.65	2.85	6.93	8	46.67	81.46
$\alpha = 0.05$	51.10	2.46	70.63	7	31.26	70.35
$\beta = 0.015$	42.57	1.97	51.99	8	23.11	56.24
$\tau = 0.45$	45.18	2.28	68.92	5	59.41	65.06
$\varepsilon = 0.04$	48.15	2.33	73.24	8	28.00	58.34

Table 1: **Comparative Statics:** Optimal leverage (%), coupon, credit spread (b.p), optimal maturity (years), tax advantage to debt (b.p), and no tax benefits-triggering firm value (Ad), for different parameter values.

represent not only corporate, but also personal taxes (Miller, 1977). Finally, the EBIT ratio generates a "price-earnings ratio" for an optimally levered firm equal to 16.25,[15] close to its historical average which is around 17.

We summary simulation results in Tables 1 and 2. In the base case the optimal leverage is 43.10 percent. Rajan and Zingalides (1995) find non equity liabilities to represent on average 44 percent of total assets for U.S firms. Optimal maturity on the other hand is 6 years, consistent with average debt maturities reported by Stohs and Mauer (1996). The tax advantage to debt is 25.8 basis points. In this, and in the rest of cases, the advantage to debt appears higher than predicted by the one period model in KMM (1985). The model is also consistent in predicting (for reasonable parameters) firm values that trigger loss of tax deductions always higher than those that trigger default. The credit spread for the base case is 51.94 basis points. Leland (1994) argues that the historical credit spread of investment-grade bonds with no call provision would be around 52 basis points.

Higher risk free interest rate seems to imply lower leverage. This is a reasonable result that optimal capital structure models have traditionally failed to generate. What distinguish our model of those is the recognition that tax benefits are lost when the firm incur in zero or negative profits. Higher risk free interest rate means higher coupon payments and higher stream of

[15] This is computed as the ratio $\frac{S}{\varepsilon A}$. Note that even though coupon payments are taken into account for taxational purposes, they actually do not alter the earnings per share given that they are already included in the equity price.

	Ab_{T-1}	Ab_{T-2}	Ab_{T-3}	Ab_{T-4}
Base Case	33.95	34.37	35.09	36.19
$r = 0.06$	33.26	34.03	35.35	40.35
$\sigma = 0.1$	67.39	67.54	67.70	67.97
$\alpha = 0.05$	40.70	41.08	41.51	42.25
$\beta = 0.015$	32.75	33.12	33.49	33.92
$\tau = 0.45$	38.55	39.28	40.44	42.21
$\varepsilon = 0.04$	37.70	38.08	38.43	38.86
	Ab_{T-5}	Ab_{T-6}	Ab_{T-7}	Ab_{T-8}
Base Case	38.00	45.10	–	–
$r = 0.06$	–	–	–	–
$\sigma = 0.1$	68.24	68.48	69.18	71.50
$\alpha = 0.05$	43.58	45.44	53.56	–
$\beta = 0.015$	34.63	35.69	37.52	44.54
$\tau = 0.45$	49.71	–	–	–
$\varepsilon = 0.04$	39.62	40.88	42.75	50.49

Table 2: **Comparative Statics (continuation):** Bankruptcy-triggering firm value when k periods remain to maturity (Ab_k), for different parameter values.

tax benefits.[16] But higher coupon also means lower probability of getting these benefits. This second effect more than offsets the first one and the firm reduces its leverage to reduce the coupon (that still will be higher than in the base case), and with it the probability of getting losses. An increase in the risk free rate has also the effect of reducing the optimal debt maturity. Lower maturity could be interpreted as a complementary policy to the reduction in leverage, that is, lower maturity also means lower probability of getting losses during the holding period. Both, the reduction in leverage and maturity, lead to a lower credit spread. Finally, the tax advantage to debt increases with the risk free rate. This reflects in some sense that "buying tax benefits becomes cheaper", said in other words, the firm would need a lower principal to obtain the same coupon, and lower principal means lower issuance costs (even though these take place more frequently).

Higher volatility of the firm unlevered assets leads to a lower leverage and a lower debt maturity. Again the consideration of positive profits as a condition for tax benefits to take place, implies different results with respect to previous works. Fischer, Heinkel and Zechner (1989) also predict that a higher volatility will reduce the initial optimal leverage ratio (in order to reduce the default probability). The coupon however will still be higher in their model because of the higher risk, and with it the tax advantage to debt. In our case, the reduction in leverage is high enough to reduce the coupon in spite of the higher firm risk. Basically, the firm searches to reduce the coupon to control the risk of falling in losses. Lower leverage implies lower issuance costs, while a higher risk results in higher benefits from allowing the firm to rebalance its leverage more often. Both of these effects bring a reduction in the maturity of the debt. The lower stream of tax benefits has a higher impact than the reduction in issuance costs and the result is a lower tax advantage to debt.

The fourth line in Table 1 analyzes the case of lower bankruptcy costs. These imply as expected a higher leverage. Lower "loss given default", allows the firm to incur in a higher default probability to increase the stream of tax benefits. Higher leverage, on the other hand, induces the firm to increase the debt maturity in order to face these cost less often. More leverage and longer maturity more than offsets the reduction in bankruptcy costs, and

[16]This is what induce for instance Leland (1994) and Fisher, Heinkel and Zechner (1989) models to predict a positive relationship between risk free interest rates and optimal initial leverage ratios.

the result is a higher credit spread. The mentioned reduction in bankruptcy costs finally brings, also as expected, an increase in the tax advantage to debt.

Next line in Table 1 considers this time a higher issuance cost. Although we observe a reduction in the leverage ratio, this is not as large in magnitude as we may presume. The higher effect is on the debt maturity: Consistent with KMM (1985), higher issuance costs makes the firm to reduce the frequency in which the firm faces these costs, and reduces the tax advantage to debt.

An increase in the relevant tax ratio on the other hand, lead to a higher leverage, increasing the credit spread. This time tax deductions are more valuable, but in principle less likely because of the higher coupon. To compensate the negative effect of a higher coupon on the probability of getting tax benefits, the firm reduces the debt maturity. Higher tax rate implies the predictable result of a higher benefit from issuing debt.

Finally, the higher the earnings ratio, the higher the leverage and the debt maturity. This is reasonable because the firm can face a higher coupon and a longer maturity while controlling the risk for non positive profits. Of course all of this implies higher tax benefits.

5 Conclusions

We have presented a dynamic model of optimal capital structure. Leverage and maturity are chosen according to the risk free interest rate, the firm risk, bankruptcy and issuance costs, tax benefits of debt, and the earnings ratio. The model considers the possibility of issuing new debt optimally at maturity of current debt, and links the availability of tax benefits to debt to the presence of taxable income. Two elements of our model introduce considerable advantages with respect to previous works: First, debt is allowed to consist in a regular coupon bond. Second, we distinguish the event of getting negative profits from the much more extreme event of default. By doing this we solve some of the limitations in Kane, Marcus and McDonald (1985) zero coupon bond model, and in Fischer, Heinkel and Zechner (1989) perpetual debt model. A numerical algorithm has been used to solve the optimization problem for different parameter values. The model shows to be able to replicate standard leverage ratios, debt maturities and credit spreads, for reasonable parameter values.

6 Appendix

6.1 Appendix 1

Here we show that even a traditional static model of the and Brennan and Schwartz (1978) type, leads to finite optimal maturity when debt is assumed to consist in a zero coupon bond.

In this context, there is a distinction between the market value of an unlevered firm, A, and that of its levered counterpart, v. By levering the firm its market value increases due to tax benefits on the payment of interests, but at the same time there is a possibility of default with its associated costs that pushes the firm value in the opposite direction. Leverage and maturity are then chosen as to maximize the firm value, that is, making the difference between tax benefits and bankruptcy costs as large as possible.

Assume the value of the firm unlevered assets evolves according to (1). This time the unlevered firm is assumed to generate a fear return, and therefore the value of any asset, F, whose value depends on A and time to maturity t, will satisfy the differential equation[17]

$$\frac{1}{2}\sigma^2 A^2 F_{AA} + rAF_A - F_t - rF = 0$$

Consider now that the firm issues a bond with maturity T and face value P. Interests are paid at maturity and are equal to c. For simplicity, and in line with Brennan and Schwartz (1978), we also assume that tax deductions, τ, apply independently of whether or not the firm has positive profits. On the other hand, if the firm defaults, its assets loose a fraction α of its market value. For any $t \geq 0$, the firm value will be the sum of the firm unlevered assets, plus the tax benefits to debt, $TB\left(A,t\right)$, less bankruptcy costs $BC\left(A,t\right)$,[18] that is

$$v\left(A,t\right) = A + TB\left(A,t\right) - BC\left(A,t\right)$$

[17]Note that under this approach it is implicitly assumed that the firm unlevered assets are still traded once the firm has been already levered. This is an important weakness of these group of models.

[18]See Leland (1994).

$TB(A,t)$ can be seen as an independent asset that provides the tax benefits generated by the firm. The boundary conditions for this asset are

$$TB(A,0) = \begin{cases} \tau c & if \quad A > P + c(1-\tau) \\ 0 & if \quad A \leq P + c(1-\tau) \end{cases}$$

And it is straightforward to show that

$$TB(A,T) = \tau c e^{-rT} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{A}{P+c(1-\tau)}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

On the other hand, $BC(A,t)$ can be seen as another independent asset with boundary conditions

$$BC(A,0) = \begin{cases} 0 & if \quad A > P + c(1-\tau) \\ \alpha A & if \quad A \leq P + c(1-\tau) \end{cases}$$

Then

$$BC(A,T) = \alpha A\left[1 - N(d_1)\right]$$

As a result

$$v(A,T) = A + \tau c e^{-rT} N(d_2) - \alpha A\left[1 - N(d_1)\right]$$

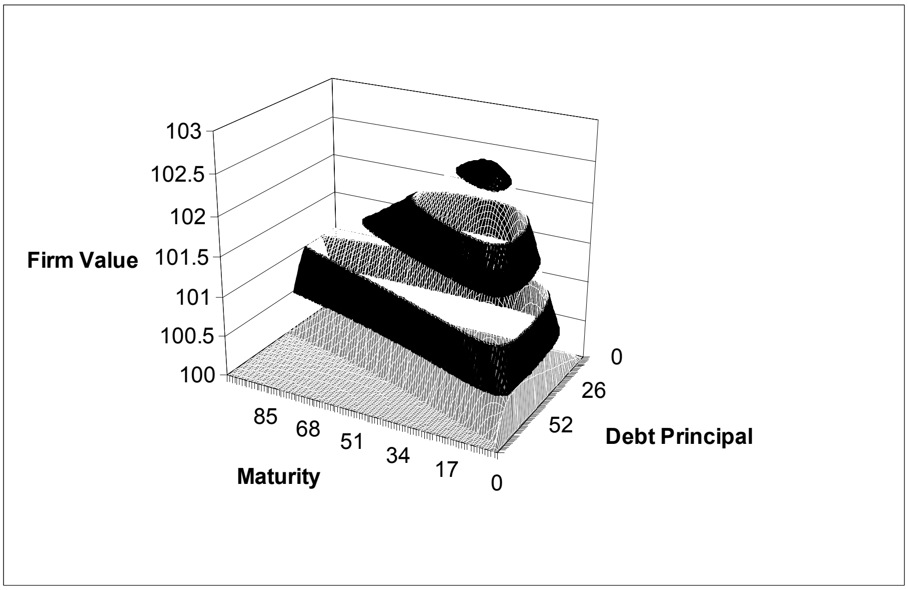

Figure 2: TB, BC and FVI as a function of P and T. Parameters are $A = 100$, $r = 0.04$, $\sigma = 0.25$, $\alpha = 0.15$ and $\tau = 0.25$.

At the same time, we could find the debt value, $D(A, T)$, by considering its boundary conditions:

$$D(A, 0) = \begin{cases} P + c & if \quad A > P + c(1 - \tau) \\ (1 - \alpha) A & if \quad A \leq P + c(1 - \tau) \end{cases}$$

What implies that

$$D(A, T) = (1 - \alpha) A [1 - N(d_1)] + (P + c) e^{-rT} N(d_2)$$

Figure 2 represents $v(A, T)$ as a function of P and T when debt is issued at par, that is, for $D(A, T) = P$.[19]

[19]Parameters are the those used in the calibration of our own model.

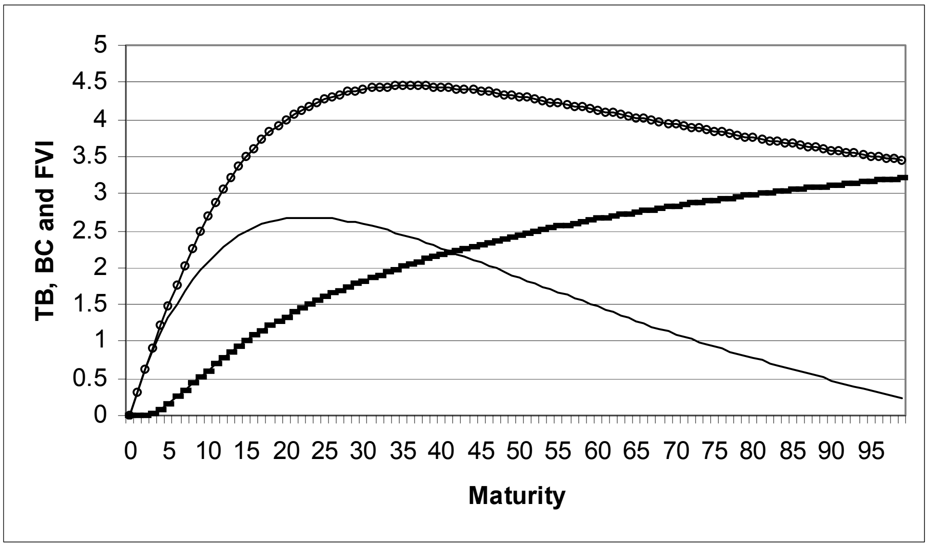

Figure 3: TB (dotted line), BC (dashed line) and FVI (solid line) as a function of T for $P = 32$. Parameters are $A = 100$, $r = 0.04$, $\sigma = 0.25$, $\alpha = 0.15$ and $\tau = 0.25$.

It is clear that there exists a global maximum, that is, a zero coupon bond model results in optimal finite maturity even in the traditional Brennan and Schwartz (1978) framework. In this case $P^* = 32$ and $T^* = 23$ (we have simply plotted $v(A, T)$ for $P = 0, 1, ..., 70$ and $T = 0, 1, ..., 99$ and have chosen the pair that yields the highest v). The firm value is increased by a 2.68 % thanks to this optimal leverage. Figure 3 represents TB, BC and the increment in firm value (FVI) as a function of the debt maturity for $P = 32$, while Figures 4 and 5 explain why we get such results: The interest payment required by debtholders increases exponentially with the maturity date (Figure 4), and the result is that the risk neutral default probability (Figure 5) tends to 1. For T large enough, the increment in the default probability more than compensates the increment in the interest payment (and the corresponding nominal tax benefit), and the present value of tax benefits starts to fall. On the other hand, as the default probability grows, the present value of bankruptcy costs also does.

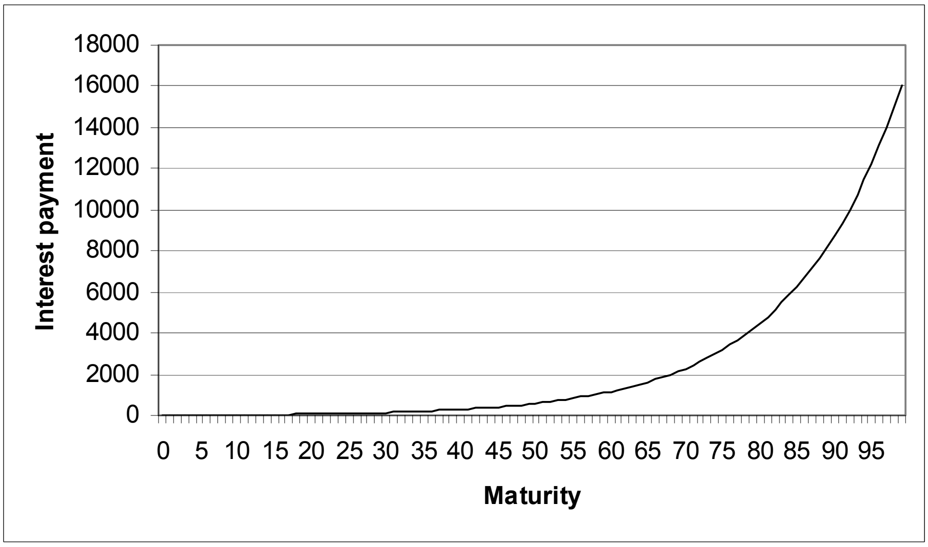

Figure 4: Interest payment as a function of T for $P = 32$. Parameters are $A = 100$, $r = 0.04$, $\sigma = 0.25$, $\alpha = 0.15$ and $\tau = 0.25$.

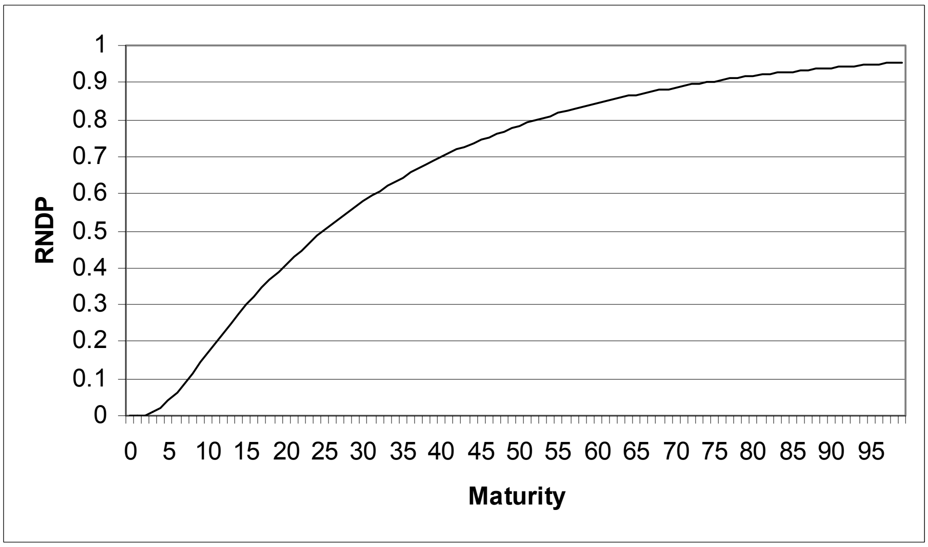

Figure 5: $RNDP$ as a function of T for $P = 32$. Parameters are $A = 100$, $r = 0.04$, $\sigma = 0.25$, $\alpha = 0.15$ and $\tau = 0.25$.

6.2 Appendix 2

We provide a formal proof of the valuation formulas presented in the core of the article. We start by computing three fundamental multiple integrals. Take $0 \leq k < t \leq T$, where k, t and T are natural numbers, and define

$$G(t,k) = e^{-r(t-k)} \int_{Ab_{t-1}}^{\infty} \int_{Ab_{t-2}}^{\infty} \dots \int_{Ab_k}^{\infty} A_k \prod_{h=k}^{t-1} f(A_h \mid A_{h+1})\, dA_k \dots dA_{t-2} dA_{t-1}$$

where $Ab_{t-1}, Ab_{t-2}, \dots, Ab_k$ are for the moment some given values, and

$$f(A_h \mid A_{h+1}) = \frac{1}{\sqrt{2\pi}\sigma A_h} \exp\left\{ -\frac{1}{2} \frac{\left\{ \ln(A_h) - \left[\ln(A_{h+1}) + \left(r - \delta - \frac{\sigma^2}{2} \right) \right] \right\}^2}{\sigma^2} \right\}$$

is the density function of A_h conditional on A_{h+1}.

Consider the following change of variable

$$\tilde{x}_l = \ln(A_l)$$

$$l = k, \dots, t$$

and define

$$f(\tilde{x}_h \mid \tilde{x}_{h+1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2} \frac{\left\{ \tilde{x}_h - \left[\tilde{x}_{h+1} + \left(r - \delta - \frac{\sigma^2}{2} \right) \right] \right\}^2}{\sigma^2} \right\}$$

Then

$$G(t,k) = e^{-r(t-k)} \int_{\ln(Ab_{t-1})}^{\infty} \int_{\ln(Ab_{t-2})}^{\infty} \dots \int_{\ln(Ab_k)}^{\infty} e^{\tilde{x}_k} \prod_{h=k}^{t-1} f(\tilde{x}_h \mid \tilde{x}_{h+1})\, d\tilde{x}_k \dots d\tilde{x}_{t-2} d\tilde{x}_{t-1}$$

Use now

$$\frac{\tilde{x}_h - \left[\tilde{x}_{h+1} + \left(r - \delta - \frac{\sigma^2}{2}\right)\right]}{\sigma} = \frac{\left\{\tilde{x}_h - \left[\tilde{x}_{h+1} + \left(r - \delta + \frac{\sigma^2}{2}\right)\right]\right\} + \sigma^2}{\sigma}$$

$$h = k, ..., t-1$$

to get

$$\begin{aligned} G(t,k) &= A_t e^{-\delta(t-k)} \int_{\ln(Ab_{t-1})}^{\infty} \int_{\ln(Ab_{t-2})}^{\infty} ... \int_{\ln(Ab_k)}^{\infty} \frac{1}{(2\pi)^{(t-k)/2}\,\sigma^{t-k}} \\ & \prod_{h=k}^{t-1} \exp\left\{ -\frac{1}{2} \frac{\left\{\tilde{x}_h - \left[\tilde{x}_{h+1} + \left(r - \delta + \frac{\sigma^2}{2}\right)\right]\right\}^2}{\sigma^2} \right\} d\tilde{x}_k ... d\tilde{x}_{t-2} d\tilde{x}_{t-1} \end{aligned}$$

We can make an additional change of variable

$$x_l = \frac{\tilde{x}_l - \left[\ln(A_t) + \left(r - \delta + \frac{\sigma^2}{2}\right)(t-l)\right]}{\sigma\sqrt{t-l}}$$

$$l = k, ..., t-1$$

Noting that

$$\frac{\tilde{x}_h - \left[\tilde{x}_{h+1} + \left(r - \delta + \frac{\sigma^2}{2}\right)\right]}{\sigma} = x_h\sqrt{t-h} - x_{h+1}\sqrt{t-h-1}$$

$$h = k, .., t-2$$

we may express $G(t,k)$ as

$$G(t,k) = A_t e^{-\delta(t-k)} \int_{-a_{t,t-1}}^{\infty} \int_{-a_{t,t-2}}^{\infty} \cdots \int_{-a_{t,k}}^{\infty} \frac{\sqrt{(t-k)!}}{(2\pi)^{(t-k)/2}}$$

$$\prod_{h=k}^{t-2} \exp\left\{ -\frac{1}{2} \left[x_h \sqrt{t-h} - x_{h+1}\sqrt{t-h-1} \right]^2 \right\}$$

$$\exp\left\{ -\frac{1}{2} x_{t-1}^2 \right\} dx_k \ldots dx_{t-2} dx_{t-1}$$

Define now $Q_z = \left\{ q_{ij}^z \right\}$, as the square symmetric matrix of dimension z, where

$$q_{11}^z = z$$

$$q_{ii}^z = 2(z-i+1) \quad for \quad i = 2, ..., z$$

$$q_{ij}^z = \begin{cases} -\sqrt{z-i+1}\sqrt{z-j+1} & if \quad |i-j| = 1 \\ 0 & if \quad |i-j| \geq 2 \end{cases}$$

It is possible to show that $Q_z = R_z^{-1}$. To see this define $W_z = R_z Q_z = \left\{ w_{ij}^z \right\}$, and consider the following cases:

a) $j = 1$

In this case it is easy to see that

$$w_{11}^z = \sum_{k=1}^{z} \rho_{1k}^z q_{k1}^z = \rho_{11}^z q_{11}^z + \rho_{12}^z q_{21}^z = 1$$

$$w_{i1}^z = \sum_{k=1}^{z} \rho_{ik}^z q_{k1}^z = \rho_{i1}^z q_{11}^z + \rho_{i2}^z q_{21}^z = 0 \quad for \quad i = 2, ..., z$$

b) $j = z$

Check now that

$$\begin{aligned} w_{iz}^{z} &= \sum_{k=1}^{z} \rho_{ik}^{z} q_{kz}^{z} = \rho_{i(z-1)}^{z} q_{(z-1)z}^{z} + \rho_{iz}^{z} q_{zz}^{z} = 0 \quad for \quad i = 1, ..., z-1 \\ w_{zz}^{z} &= \sum_{k=1}^{z} \rho_{zk}^{z} q_{kz}^{z} = \rho_{z(z-1)}^{z} q_{(z-1)z}^{z} + \rho_{zz}^{z} q_{zz}^{z} = 1 \end{aligned}$$

c) $1 < j < z$

In this case

$$w_{ij}^{z} = \sum_{k=1}^{z} \rho_{ik}^{z} q_{kj}^{z} = \rho_{i(j-1)}^{z} q_{(j-1)j}^{z} + \rho_{ij}^{z} q_{jj}^{z} + \rho_{i(j+1)}^{z} q_{(j+1)j}^{z}$$

Consider the three possible situations, namely $i \leq j-1$, $i = j$, and $i \geq j+1$, and make straightforward computations to see that $w_{ii}^{z} = 1$ and $w_{ij}^{z} = 0$ for $i \neq j$; $i = 1, ..., z$; $j = 2, ..., z-1$.

We can now use previous arguments, and $|R_z| = \frac{1}{z!}$, to express $G(t, k)$ as

$$\begin{aligned} G(t,k) &= A_t e^{-\delta(t-k)} \int_{-a_{t,t-1}}^{\infty} \int_{-a_{t,t-2}}^{\infty} ... \int_{-a_{t,k}}^{\infty} \frac{1}{(2\pi)^{(t-k)/2} \left|R_{t-k}\right|^{1/2}} \\ &\exp\left\{-\frac{1}{2} X R_{t-k}^{-1} X'\right\} dx_k ... dx_{t-2} dx_{t-1} \end{aligned}$$

where $X = [x_k \ ... \ x_{t-2} \ x_{t-1}]$. Finally

$$G(t,k) = A_t e^{-\delta(t-k)} N_{t-k}(\mathbf{a}_{t,k})$$
$$t = 1, ..., T$$
$$k = 0, ..., t-1$$

Define also

$$\hat{G}(t,k) = e^{-r(t-k)} \int_{Ab_{t-1}}^{\infty} \int_{Ab_{t-2}}^{\infty} ... \int_{0}^{Ab_k} A_k \prod_{h=k}^{t-1} f(A_h \mid A_{h+1})\, dA_k ... dA_{t-2} dA_{t-1}$$

Previous derivations imply that

$$\begin{aligned} \hat{G}(t,k) &= A_t e^{-\delta(t-k)} \int_{-a_{t,t-1}}^{\infty} \int_{-a_{t,t-2}}^{\infty} ... \int_{-\infty}^{-a_{t,k}} \frac{1}{(2\pi)^{(t-k)/2} \left|R_{t-k}\right|^{1/2}} \\ & \exp\left\{-\frac{1}{2} X R_{t-k}^{-1} X'\right\} dx_k ... dx_{t-2} dx_{t-1} \end{aligned}$$

and finally[20]

$$\hat{G}(t,k) = A_t e^{-\delta(t-k)} \left[N_{t-(k+1)}(\mathbf{a}_{t,k+1}) - N_{t-k}(\mathbf{a}_{t,k})\right]$$
$$t = 1, ..., T \qquad (18)$$
$$k = 0, ..., t-1$$

[20] An alternative approach to evaluate multiple integrals like $\hat{G}(t,k)$ using multivariate normal cumulative distributions functions, is to consider an appropriate transformation of the correlation matrix. An example is in the valuation formula for American put options derived by Geske and Johnson (1984).

Another useful multiple integral is the following

$$H(t,k) = e^{-r(t-k)} \int_{Ab_{t-1}}^{\infty} \int_{Ab_{t-2}}^{\infty} ... \int_{Ab_k}^{\infty} \prod_{h=k}^{t-1} f(A_h \mid A_{h+1})\, dA_k ... dA_{t-2} dA_{t-1}$$

Define

$$y_l = \frac{\ln(A_l) - \left[\ln(A_t) + \left(r - \delta - \frac{\sigma^2}{2}\right)(t-l)\right]}{\sigma\sqrt{t-l}}$$

$$l = k, ..., t-1$$

then

$$\frac{\ln(A_h) - \left[\ln(A_{h+1}) + \left(r - \delta - \frac{\sigma^2}{2}\right)\right]}{\sigma} = y_h\sqrt{t-h} - y_{h+1}\sqrt{t-h-1}$$

$$h = k, ..., t-2$$

and $H(t,k)$ reduces to

$$H(t,k) = e^{-r(t-k)} \int_{-b_{t,t-1}}^{\infty} \int_{-b_{t,t-2}}^{\infty} ... \int_{-b_{t,k}}^{\infty} \frac{1}{(2\pi)^{(t-k)/2} \left|R_{t-k}\right|^{1/2}}$$

$$\exp\left\{-\frac{1}{2} Y R_{t-k}^{-1} Y'\right\} dy_k ... dy_{t-2} dy_{t-1}$$

where $Y = [y_k \; ... \; y_{t-2} \; y_{t-1}]$. If we also set $H(t,k) = 1$ for $t = k$, then the result is that

$$H(t,k) = e^{-r(t-k)} N_{t-k}(\mathbf{b}_{t,k})$$

$$t = 1, ..., T$$

$$k = 0, ..., t$$

A final multiple integral we will be using is

$$I\left(t,k\right)=\begin{cases} if \quad t-k=1 \\ e^{-r}\int_{Ad}^{\infty}f\left(A_{k}\mid A_{k+1}\right)dA_{k} \\ and \quad if \quad t-k\geq 2 \\ e^{-r(t-k)}\int_{Ab_{t-1}}^{\infty}...\int_{Ab_{k+1}}^{\infty}\int_{Ad}^{\infty}\prod_{h=k}^{t-1}f\left(A_{h}\mid A_{h+1}\right)dA_{k}dA_{k+1}...dA_{t-1}\end{cases}$$

If we now set $I\left(t,k\right)=1$ for $t=k$, then previous arguments result in

$$I\left(t,k\right)=e^{-r(t-k)}N_{t-k}\left(\mathbf{c}_{t,k}\right)$$

$$t=1,...,T$$

$$k=0,...,t$$

At this point we are ready to derive specific expressions for equity, debt, and firm value.

Equity:

The equity value when the debt matures will be given by

$$S\left(A,0\right)=\begin{cases} A-P-\left(1-\tau\right)c & if \quad A>Ad \\ Max\left\{0,A-P-c\right\} & if \quad A\leq Ad\end{cases}$$

and the firm will default whenever $A_{0}\leq Ab_{0}=P+c$.

When only one year remains to maturity

$$S(A,1)=\begin{cases} G(1,0)-PH(1,0)-\sum_{k=0}^{1}c\left[H(1,k)-\tau I(1,k)\right] & if \quad A>Ad \\ Max\left\{0,G(1,0)-PH(1,0)\right. & \\ \qquad\qquad \left.-c\left[H(1,0)-\tau I(1,0)\right]-c\right\} & if \quad A\leq Ad \end{cases}$$

$$=\begin{cases} Ae^{-\delta}N_1\left(\mathbf{a}_{1,0}\right)-Pe^{-r}N_1\left(\mathbf{b}_{1,0}\right) & \\ \quad -\sum_{k=0}^{1}ce^{-r(1-k)}\left[N_{1-k}\left(\mathbf{b}_{1,k}\right)-\tau N_{1-k}\left(\mathbf{c}_{1,k}\right)\right] & if \quad A>Ad \\ Max\left\{Ae^{-\delta}N_1\left(\mathbf{a}_{1,0}\right)-Pe^{-r}N_1\left(\mathbf{b}_{1,0}\right)\right. & \\ \qquad \left.ce^{-r}\left[N_1\left(\mathbf{b}_{1,0}\right)-\tau N_1\left(\mathbf{c}_{1,0}\right)\right]-c\right\} & if \quad A\leq Ad \end{cases}$$

The firm will default this time whenever $A_1 \leq Ab_1$, but now the default threshold Ab_1 will be some non explicit value. Solving for Ab_1 leads to

$$S(A,2)=\begin{cases} G(2,0)-PH(2,0)-\sum_{k=0}^{2}c\left[H(2,k)-\tau I(2,k)\right] & if \quad A>Ad \\ Max\left\{0,G(2,0)-PH(2,0)\right. & \\ \qquad\qquad \left.-\sum_{k=0}^{1}c\left[H(2,k)-\tau I(2,k)\right]-c\right\} & if \quad A\leq Ad \end{cases}$$

$$
= \begin{cases} Ae^{-\delta 2} N_2\left(\mathbf{a}_{2,0}\right) - Pe^{-r2} N_2\left(\mathbf{b}_{2,0}\right) \\ \qquad\qquad - \sum_{k=0}^{2} ce^{-r(2-k)}\left[N_{2-k}\left(\mathbf{b}_{2,k}\right) - \tau N_{2-k}\left(\mathbf{c}_{2,k}\right)\right] & if \quad A > Ad \\ Max\left\{0, Ae^{-\delta 2} N_2\left(\mathbf{a}_{2,0}\right) - Pe^{-r2} N_2\left(\mathbf{b}_{2,0}\right)\right. \\ \qquad \left. - \sum_{k=0}^{1} ce^{-r(2-k)}\left[N_{2-k}\left(\mathbf{b}_{2,k}\right) - \tau N_{2-k}\left(\mathbf{c}_{2,k}\right)\right] - c\right\} & if \quad A \leq Ad \end{cases}
$$

In the same way we can find $S(A,t)$, $t = 3, ..., T^+$, and $S^-(A,T)$ as $S^+(A,T) - \beta P$.

Firm:

We may now compute both, $V(A,t)$ and $TB(A,t)$. First note that

$$
V(A,0) = \begin{cases} A & if \quad A > Ab_0 \\ (1-\alpha) A & if \quad A \leq Ab_0 \end{cases}
$$

and then

$$
V(A,1) = \begin{cases} G(1,0) + (1-\alpha)\hat{G}(1,0) & if \quad A > Ab_1 \\ (1-\alpha) A & if \quad A \leq Ab_1 \end{cases}
$$

$$
= \begin{cases} Ae^{-\delta} N_1\left(\mathbf{a}_{1,0}\right) + (1-\alpha) Ae^{-\delta}\left[1 - N_1\left(\mathbf{a}_{1,0}\right)\right] & if \quad A > Ab_1 \\ (1-\alpha) A & if \quad A \leq Ab_1 \end{cases}
$$

For $t = 2$

$$V(A,2)=\begin{cases} G(2,0)+(1-\alpha)\sum_{k=0}^{1}\hat{G}(2,k) & if \quad A>Ab_2 \\ (1-\alpha)A & if \quad A\leq Ab_2 \end{cases}$$

$$=\begin{cases} Ae^{-\delta 2}N_2\left(\mathbf{a}_{2,0}\right) & \\ +(1-\alpha)A\sum_{k=0}^{1}e^{-\delta(2-k)}\left[N_{2-(k+1)}\left(\mathbf{a}_{2,k+1}\right)-N_{2-k}\left(\mathbf{a}_{2,k}\right)\right] & if \quad A>Ab_2 \\ (1-\alpha)A & if \quad A\leq Ab_2 \end{cases}$$

and in general, for any $t<T$

$$V(A,t)=\begin{cases} G(t,0)+(1-\alpha)\sum_{k=0}^{t-1}\hat{G}(t,k) & if \quad A>Ab_t \\ (1-\alpha)A & if \quad A\leq Ab_t \end{cases}$$

$$=\begin{cases} Ae^{-\delta t}N_t\left(\mathbf{a}_{t,0}\right) & \\ +(1-\alpha)A\sum_{k=0}^{t-1}e^{-\delta(t-k)}\left[N_{t-(k+1)}\left(\mathbf{a}_{t,k+1}\right)-N_{t-k}\left(\mathbf{a}_{t,k}\right)\right] & if \quad A>Ab_t \\ (1-\alpha)A & if \quad A\leq Ab_t \end{cases}$$

while for $t=T$

$$V(A,T)=G(T,0)+(1-\alpha)\sum_{k=0}^{T-1}\hat{G}(t,k)$$

what drive us to expression (4).

On the other hand

$$TB\left(A,0\right)=\begin{cases}\tau c \quad if \quad A>Ad\\ \\ 0 \quad if \quad A\leq Ad\end{cases}$$

then

$$TB\left(A,1\right)=\begin{cases}\sum_{k=0}^{1}\tau cI(1,k) & if & A>Ad\\ \\ \tau cI(1,0) & if & Ab_1<A\leq Ad\\ \\ 0 & if & A\leq Ab_1\end{cases}$$

$$=\begin{cases}\sum_{k=0}^{1}\tau ce^{-r(1-k)}N_{1-k}\left(\mathbf{c}_{1,k}\right) & if & A>Ad\\ \\ \tau ce^{-r}N_1\left(\mathbf{c}_{1,0}\right) & if & Ab_1<A\leq Ad\\ \\ 0 & if & A\leq Ab_1\end{cases}$$

and in general

$$TB\left(A,t\right)=\begin{cases}\sum_{k=0}^{t}\tau cI(t,k) & if & A>Ad\\ \\ \sum_{k=0}^{t-1}\tau cI(t,k) & if & Ab_t<A\leq Ad\\ \\ 0 & if & A\leq Ab_t\end{cases}$$

$$
= \begin{cases} \sum_{k=0}^{t} \tau c e^{-r(t-k)} N_{t-k}\left(\mathbf{c}_{t,k}\right) & if \qquad A > Ad \\ \\ \sum_{k=0}^{t-1} \tau c e^{-r(t-k)} N_{t-k}\left(\mathbf{c}_{t,k}\right) & if \quad Ab_t < A \leq Ad \\ \\ 0 & if \qquad A \leq Ab_t \end{cases}
$$

for $t < T$, while $TB(A,T)$ results in expression (5).

Finally, $v(A,t) = V(A,t) + TB(A,t)$ for $t = 0, ..., T^+$, whereas $v^-(A,T)$ will be given by (6).

Debt:

The debt value will be the firm value minus the equity value

For any $t < T$

$$
D(A,t) = \begin{cases} (1-\alpha) A \sum_{k=0}^{t-1} e^{-\delta(t-k)} \left[N_{t-(k+1)}\left(\mathbf{a}_{t,k+1}\right) - N_{t-k}\left(\mathbf{a}_{t,k}\right)\right] & \\ \\ +Pe^{-rt} N_t\left(\mathbf{b}_{t,0}\right) + \sum_{k=0}^{t} c e^{-r(t-k)} N_{t-k}\left(\mathbf{b}_{t,k}\right) & if \quad A > Ab_t \\ \\ (1-\alpha) A & if \quad A \leq Ab_t \end{cases}
$$

while $D(A,T)$ is given in (7).■

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